Let A and B be sets and $f : A \to B$ be some function. Note: If $A = \emptyset$, then $f : \emptyset \to B$ must send nothing nowhere. This is the only function with domain $\emptyset$ and codomain B. It is called the **empty function**. If $B = \emptyset$ and $A \neq \emptyset$, then there is no function $f : A \to B$ (since the elements of A have nowhere to go). Thus a function whose codomain is empty must also have an empty domain. Confusing but true. For what follows we will *assume that* A *and* B *are non-empty.*

Recall that $\mathrm{id}_A : A \to A$ is the **identity function** where $\mathrm{id}_A(x) = x$ for all $x \in A$. We say that f has a **left inverse** $g : B \to A$ if $g \circ f = \mathrm{id}_A$ and f has a right inverse $h : B \to A$ if $f \circ h = \mathrm{id}_B$. Since function composition is associative, if f has a left inverse g and right inverse h, then $g = g \circ \mathrm{id}_B = g \circ (f \circ h) = (g \circ f) \circ h = \mathrm{id}_A \circ h = h$. Thus if f has both a left and right inverse, they must match. We call such a two-sided inverse (which we just proved is unique) *the* **inverse** of f and denote it by f^{-1}.

Proposition: Let $f : A \to B$ (where A and B are non-empty).

i) The function f has a left inverse if and only if it is injective (i.e., one-to-one).

ii) The function f has a right inverse if and only if it is surjective (i.e., onto).

iii) The function f has an inverse if and only if it is bijective (i.e., one-to-one and onto).

Proof:

i) Suppose f has a left inverse $g : B \to A$ and suppose that $f(a) = f(b)$ for some $a, b \in A$. Then $a = \mathrm{id}_A(a) = (g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ f)(b) = \mathrm{id}_B(b) = b$. Thus f is injective.

   Now suppose f is injective. Pick some $a_0 \in A$ (A is non-empty so this is possible). Next, consider any element y in the range of f. Then there is some $x \in A$ such that $f(x) = y$ (this is what it means to belong to the range of f). Notice that if $f(x) = y = f(x')$ then $x = x'$ since f is one-to-one. Therefore, for any element y in the range of f, there is a *unique* element $x \in A$ such that $f(x) = y$. Call this element $x = f^{-1}(y)$. Now define $g : B \to A$ as follows:
$$g(y) = \begin{cases} f^{-1}(y) & y \text{ is in the range of } f \\ a_0 & y \text{ is not in the range of } f \end{cases}$$
   Finally, notice that $g(f(x)) = f^{-1}(f(x)) = x$ where we use the first formula in g since $f(x)$ belongs to the range of f and $f^{-1}(f(x)) = x$ by the way we defined what "$f^{-1}(y)$" means (it is the element that maps to y by f – certainly x is the element that maps to $f(x)$ by f). Therefore, $g \circ f = \mathrm{id}_A$ so that g is a left inverse for f.

ii) Suppose that f has a right inverse $h : B \to A$ and suppose $y \in B$. Then $h(y) \in A$ and $f(h(y)) = \mathrm{id}_B(y) = y$. Therefore, y is in the range of f and thus f is surjective (i.e., onto).

   Now suppose f is surjective. For each $b \in B$, we know that b is in the range of f (because it's onto). Thus we can choose some $a \in A$ such that $f(a) = b$. We (arbitrarily) choose such an input for each b. Let's name our choice a_b (so $a_b \in A$ and $f(a_b) = b$). Now define $h : B \to A$ as follows: $h(b) = a_b$. Since we chose exactly one a_b for each $b \in B$, we have a well defined function h. Then $(f \circ h)(y) = f(h(y)) = f(a_y) = y = \mathrm{id}_B(y)$ for all $y \in B$. Therefore, $f \circ h = \mathrm{id}_B$ and thus h is a right inverse of f.

iii) We now know that f has an inverse if and only if it has a left and right inverse if and only if it is both injective and surjective if and only if it is bijective.

■

Without proof I will state the following (easy to prove) proposition:

Proposition: Let $f : A \to B$ and $g : B \to C$.

i) If both f and g are injective, then $g \circ f$ is too.

ii) If both f and g are surjective, then $g \circ f$ is too.

iii) If both f and g are bijective, then $g \circ f$ is too. Moreover, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition: Let $f : A \to B$. Then we *induce* two functions on the powersets of A and B.

- $f : \mathcal{P}(A) \to \mathcal{P}(B)$ where given any $C \subseteq A$ we define $f(C) = \{f(x) \mid x \in C\}$. We call $f(A)$ the **image** of C under the function f.
- $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ where given any $D \subseteq B$ we define $f^{-1}(D) = \{x \in A \mid f(x) \in D\}$. We call $f^{-1}(D)$ the **inverse image** or **preimage** of D under the function f.

Let me emphasize: The preimage is defined regardless of whether f has an inverse or not! Keep in mind that by definition: $x \in f^{-1}(D)$ if and only if $f(x) \in D$.

The first thing to acknowledge is that many people find this notation confusing. This is a classical *abuse of notation*. We are using f for both the name of the original function and for a new function between powersets. Also, f^{-1} is being used as a new function between powersets when the original function may not even have an inverse! In fact, if f does have an inverse (i.e., it is bijective), then $f^{-1}(D)$ has two different meanings. It could mean the image of D under f^{-1} or the preimage of D under f. In the end, this is ok since in the case that f has an inverse, these sets are the same.

Quick proof: Suppose f is invertible. Let $D \subseteq B$, let M be the image of D under f^{-1} and N be the preimage of D under f. Suppose $x \in M$. Then because M is the image of D under f^{-1}, there exists some $y \in D$ such that $f^{-1}(y) = x$. Therefore, $y = f(x)$ and so $x \in N$ since $f(x) = y \in D$. Conversely, let $x \in N$. Then by definition, $f(x) \in D$ and so $x = f^{-1}(f(x))$ is the image under f^{-1} of the element $f(x)$ in D. Thus $x \in M$. Therefore, $x \in M$ if and only if $x \in N$ (i.e., $M = N$).

The range of a function is the set of all outputs. This is the same as the image of the domain. The range is thus denoted in a bunch of different ways: $\text{range}(f) = \text{image}(f) = f(A)$ (and others).

Example: Consider the function $f : \{a, b, c\} \to \{1, 2, 3, 4\}$ where $f(a) = 1$, $f(b) = 2$, and $f(c) = 2$. Then $f(\{a, c\}) = \{f(a), f(c)\} = \{1, 2\}$. Also, $f(\{a, b, c\}) = \{f(a), f(b), f(c)\} = \{1, 2\}$. This means that the range of f is $\{1, 2\}$. We could also consider the other extreme case, $f(\emptyset) = \emptyset$ (nothing maps to nothing). On the other hand, $f^{-1}(\emptyset) = \{x \in \{a, b, c\} \mid f(x) \in \emptyset\} = \emptyset$ (only nothing maps to nothing). Consider $f^{-1}(\{2, 3, 4\}) = \{x \in \{a, b, c\} \mid f(x) \in \{2, 3, 4\}\} = \{b, c\}$ since $f(b) = f(c) = 2$ but a doesn't map to 2, 3, or 4. Again notice that $f^{-1}(\{2\}) = \{b, c\}$ and that $f^{-1}(\{3, 4\}) = \emptyset$ (nothing maps to 3 or 4).

Staring at the above example, one might make the following observations:

Proposition: Let $f : A \to B$.

i) The function f is injective (i.e., one-to-one) if and only if for every $x \in B$, $f^{-1}(\{x\})$ has at most one element.

ii) The function f is surjective (i.e., onto) if and only if for every $x \in B$, $f^{-1}(\{x\})$ is not empty.

In fact, there is a special name for the inverse image of a singleton set such as $f^{-1}(\{x\})$. We call this the **fiber** over x. Thus onto functions are exactly those with non-empty fibers and one-to-one functions are exactly those whose non-empty fibers have exactly one element. Having a non-empty fiber over x means that x is in the range of our function.

One could ask if $f : \mathcal{P}(A) \to \mathcal{P}(B)$ and $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ are inverses – the notation would seem to indicate that they are. But, alas, in this case we are victims of bad notation. I will state without proof (although the proofs are not difficult) the following result:

Proposition: Let $f : A \to B$.

i) For any $C \subseteq A$, it is always the case that $C \subseteq f^{-1}(f(C))$.

ii) On the other hand, $f^{-1}(f(C)) = C$ for all $C \subseteq A$ if and only if f is injective (i.e., one-to-one).
Thus $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ is a left inverse of $f : \mathcal{P}(A) \to \mathcal{P}(B)$ if and only if $f : A \to B$ has a left inverse (i.e., it is injective = one-to-one).

iii) For any $D \subseteq B$, it is always the case that $f(f^{-1}(D)) \subseteq D$.

iv) On the other hand, $f(f^{-1}(D)) = D$ for all $D \subseteq B$ if and only if f is surjective (i.e., onto).
Thus $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ is a right inverse of $f : \mathcal{P}(A) \to \mathcal{P}(B)$ if and only if $f : A \to B$ has a right inverse (i.e., it is surjective = onto).

v) $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ is the inverse of $f : \mathcal{P}(A) \to \mathcal{P}(B)$ if and only if f has an inverse (i.e., it is bijective = one-to-one and onto).