Formal Number Theory: Proofs in E and PA

Axioms:

Axiom 1: $A \to (B \to A)$

Axiom 2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Axiom 3: $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Axiom 4: $(\forall x A(x)) \to A(t)$, provided that t is free for x in $A(x)$.

Axiom 5: $\forall x(A \to B) \to (A \to \forall x B)$, provided that x does not occur free in A.

Axiom 6: $\forall x(x = x)$.

Axiom 7: For every formula $\psi(x, y)$, with free variable x, if y is free for x in $\psi(x, x)$ then

$$x = y \to (\psi(x, x) \to \psi(x, y)).$$

Axiom 8: $\forall x \forall y(x = y \to x' = y')$

Axiom 9: $\forall x(0 \neq x')$

Axiom 10: $\forall x \forall y(x' = y' \to x = y)$

Axiom 11: $\forall x(x + 0 = x)$

Axiom 12: $\forall x \forall y(x + (y') = (x + y)')$

Axiom 13: $\forall x(x \cdot 0 = 0)$

Axiom 14: $\forall x \forall y(x \cdot (y') = (x \cdot y) + x)$

Axiom 15: If $A(x)$ is a formula of S, then $A(0) \to (\forall n(A(n) \to A(n')) \to \forall n A(n))$

Rules of Inference:

Modus Ponens (MP): From A and $A \to B$ deduce B.

Generalization (GEN): From A, deduce $\forall x A$.

Shortcuts:

The same shortcuts for L and K apply as well as abbreviations for $\vee$, $\wedge$, $\leftrightarrow$, and $\exists$.

Theorems:

E1: $\vdash_E \forall x \forall y(x = y \to y = x)$

E2: $\vdash_E (x = y \to (y = z \to x = z))$

E3: $\vdash_E (x = y \wedge x = z) \to y = z$

E4: $\vdash_E \forall x \forall y \forall z(y = x \to (z = x \to y = z)$

PA1: $\vdash_{PA} \forall n(0 \cdot n = 0 \to 0 \cdot (n') = 0)$.

PA2: $\vdash_{PA} \forall n(0 \cdot n = 0)$.

PA3: $\vdash_{PA} \forall n(n = 0 + n)$.

PA4: $\vdash_{PA} \forall x \forall y(x + y = y + x)$

PA5: $\vdash_{PA} \forall x \forall y(x \cdot y = y \cdot x)$