

Techniques for solving systems of linear equations lie at the heart of linear algebra. In high school we learn to solve systems with 2 or 3 variables using “elimination” and “substitution” of variables. In order to solve systems with a large number of variables we need to be more organized. The process of *Gauss-Jordan Elimination* gives us a systematic way of solving linear systems.

To solve a system of equations we should first drop as many unnecessary symbols as possible. This is done by constructing an *augmented matrix*.

**Example:**

$$\begin{array}{rrcrcl} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \implies \left[ \begin{array}{ccc|c} 2 & -1 & 3 & -1 \\ 0 & 5 & -6 & 0 \\ -1 & 0 & 4 & 7 \end{array} \right]$$

To solve our system we need to manipulate our equations. We will see that standard manipulations correspond to changing the *rows* of our augmented matrix. In the end, it turns out that we need just 3 types of operations to solve any linear system. We call these *elementary operations*.

### Definition: Elementary Row Operations

	Effect on the linear system:		Effect on the matrix:
<b>Type I</b>	Interchange equation $i$ and equation $j$ (List the equations in a different order.)	$\iff$	Swap Row $i$ and Row $j$
<b>Type II</b>	Multiply both sides of equation $i$ by a non-zero scalar $c$	$\iff$	Multiply Row $i$ by $c$ where $c \neq 0$
<b>Type III</b>	Multiply both sides of equation $i$ by $c$ and add to equation $j$	$\iff$	Add $c$ times Row $i$ to Row $j$ where $c$ is any scalar

If we can get matrix  $A$  from matrix  $B$  by performing a series of elementary row operations, then  $A$  and  $B$  are called **row equivalent matrices**.

### Example: Type I — swap rows 1 and 3

$$\begin{array}{rrcrcl} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \implies \begin{array}{rrcrcl} -x & & & + & 4z & = & 7 \\ & & 5y & - & 6z & = & 0 \\ 2x & - & y & + & 3z & = & -1 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & -1 & 3 & -1 \\ 0 & 5 & -6 & 0 \\ -1 & 0 & 4 & 7 \end{array} \right] \xrightarrow{R1 \leftrightarrow R3} \left[ \begin{array}{ccc|c} -1 & 0 & 4 & 7 \\ 0 & 5 & -6 & 0 \\ 2 & -1 & 3 & -1 \end{array} \right]$$

### Example: Type II — scale row 3 by -2

$$\begin{array}{rrcrcl} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \implies \begin{array}{rrcrcl} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ 2x & & & + & -8z & = & -14 \end{array}$$

$$\left[ \begin{array}{ccc|c} 2 & -1 & 3 & -1 \\ 0 & 5 & -6 & 0 \\ -1 & 0 & 4 & 7 \end{array} \right] \xrightarrow{-2 \times R3} \left[ \begin{array}{ccc|c} 2 & -1 & 3 & -1 \\ 0 & 5 & -6 & 0 \\ 2 & 0 & -8 & -14 \end{array} \right]$$



The forward pass is now complete. Now our matrix is in row echelon form (in my sense not our textbook's sense). Sometimes the forward pass alone is referred to as "Gaussian Elimination". However, we should be careful since the term "Gaussian Elimination" more commonly refers to *both* the forward and backward passes. Now let's finish Gauss-Jordan Elimination by performing a **backward pass**:

1. If necessary, scale the rightmost unfinished pivot to 1 (use a Type II operation).
2. Add appropriate multiples of the current pivot's row to rows above it so that only 0 appears above the current pivot (using several Type III operations).
3. The current pivot is now "finished". If any unfinished pivots remain, go to step 4.

It should be fairly obvious that the entire Gauss-Jordan algorithm will terminate in finitely many steps. Also, only elementary row operations have been used. So we end up with a row equivalent matrix. A tedious, wordy, and unenlightening proof would show us that the resulting matrix is in reduced row echelon form (RREF).

**Example:** Let's solve the system

$$\begin{array}{rcrcrcrcrcrcl} x & + & 2y & = & 1 \\ 3x & + & 4y & = & -1 \end{array}$$

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & -1 \end{array} \right] \xrightarrow{-3 \times R1 + R2} \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array} \right]$$

The first non-zero column is just the first column. So the upper left hand corner is a pivot position. This position already has a non-zero entry so no swap is needed. The type III operation "−3 times row 1 added to row 2" clears the only position below the pivot, so after one operation we have finished with this pivot and can ignore row 1.

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array} \right]$$

Among the (unignored parts of) columns the leftmost non-zero column is the second column. So the "−2" sits in a pivot position. Since it's non-zero, no swap is needed. Also, there's nothing below it, so no type III operations are necessary. Thus we're done with this row and we can ignore it.

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array} \right]$$

Nothing's left so we're done with the forward pass.  $\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array} \right]$  is in row echelon form.

Next, we need to take the rightmost pivot (the "−2") and scale it to 1 then clear everything above it.

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -4 \end{array} \right] \xrightarrow{-1/2 \times R2} \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array} \right] \xrightarrow{-2 \times R2 + R1} \left[ \begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array} \right]$$

This "finishes" that pivot. The next rightmost pivot is the 1 in the upper left hand corner. But it's already scaled to 1 and has nothing above it, so it's finished as well. That takes care of all of the pivots so the backward pass is complete leaving our matrix in reduced row echelon form.

Finally, let translate the RREF matrix back into a system of equations. The (new equivalent) system is 
$$\begin{array}{rcrcrcrcrcrcl} x & = & -3 \\ y & = & 2 \end{array}$$
. So the only solution for this system is  $x = -3$  and  $y = 2$ .

*Note:* One can also solve a system quite easily once (just) the forward pass is complete. This is done using "back substitution". Notice that the system after the forward pass was 
$$\begin{array}{rcrcrcrcrcrcl} x & + & 2y & = & 1 \\ -2y & = & -4 \end{array}$$
. So we have  $-2y = -4$  thus  $y = 2$ . Substituting this back into the first equation we get  $x + 2(2) = 1$  so  $x = -3$ .

**Example:** Let's solve the system

$$\begin{array}{rrcr} 2x & - & y & + & z & = & 0 \\ 4x & - & 2y & + & 2z & = & -1 \\ x & & & + & z & = & 1 \end{array}$$

$$\begin{aligned} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 4 & -2 & 2 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{array} \right] & \xrightarrow{-2 \times R1 + R2} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{array} \right] & \xrightarrow{-1/2 \times R1 + R3} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{array} \right] & \text{Ignore } R1 \\ \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{array} \right] & \xrightarrow{R2 \leftrightarrow R3} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \xrightarrow{\text{Ignore } R2} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \text{Ignore } R3 \end{aligned}$$

which leaves us with nothing. So the forward pass is complete and  $\left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{array} \right]$  is in REF.

$$\begin{aligned} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \xrightarrow{1 \times R3 + R2} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \xrightarrow{2 \times R2} \left[ \begin{array}{cccc|c} 2 & -1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \xrightarrow{1 \times R2 + R1} \\ & \left[ \begin{array}{cccc|c} 2 & 0 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{array} \right] & \xrightarrow{1/2 \times R1} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{array} \right] \end{aligned}$$

This finishes the backward pass and our matrix is now in RREF. Our new system of equations is

$$\begin{array}{rrcr} x & & & + & z & = & 0 \\ & y & + & z & = & 0 \\ & & 0 & = & 1 \end{array}$$

Of course  $0 \neq 1$ , so this is an **inconsistent** system — it has **no solutions**.

*Note:* If our only goal was to solve this system, we could have stopped after the very first operation (row number 2 already said “ $0 = 1$ ”).

**Example:** Let's solve the system

$$\begin{array}{rrcr} x & + & 2y & + & 3z & = & 3 \\ 4x & + & 5y & + & 6z & = & 9 \\ 7x & + & 8y & + & 9z & = & 15 \end{array}$$

$$\begin{aligned} \left[ \begin{array}{cccc|c} 1 & 2 & 3 & : & 3 \\ 4 & 5 & 6 & : & 9 \\ 7 & 8 & 9 & : & 15 \end{array} \right] & \xrightarrow{-4 \times R1 + R2} \left[ \begin{array}{cccc|c} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 7 & 8 & 9 & : & 15 \end{array} \right] & \xrightarrow{-7 \times R1 + R3} \left[ \begin{array}{cccc|c} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & -6 & -12 & : & -6 \end{array} \right] & \xrightarrow{-2 \times R2 + R3} \\ & \left[ \begin{array}{cccc|c} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{array} \right] & \xrightarrow{-1/3 \times R2} \left[ \begin{array}{cccc|c} 1 & 2 & 3 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{array} \right] & \xrightarrow{-2 \times R2 + R1} \left[ \begin{array}{cccc|c} 1 & 0 & -1 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{array} \right] \end{aligned}$$

Now our matrix is in RREF. The new system of equations is

$$\begin{array}{rrcr} x & & & - & z & = & 1 \\ & y & + & 2z & = & 1 \\ & & 0 & = & 0 \end{array}$$

This is new —

we don't have an equation of the form “ $z = \dots$ ” This is because  $z$  does not lie in a pivot column. So we can make  $z$  a “free variable.” Let's relabel it  $z = t$ . Then we have  $x - t = 1$ ,  $y + 2t = 1$ , and  $z = t$ . So  $x = 1 + t$ ,  $y = 1 - 2t$ , and  $z = t$  is a solution for any choice of  $t$ . In particular,  $x = y = 1$  and  $z = 0$  is a solution. But so is  $x = 2$ ,  $y = -1$ ,  $z = 1$ . In fact, there are infinitely many solutions.

*Note:* A system of linear equations will always have either one solution, infinitely many solutions, or no solution at all.

**Multiple Systems:** Gauss-Jordan can handle solving multiple systems at once, if these systems share the same coefficient matrix (the part of the matrix before the :’s).

Suppose we wanted to solve both 
$$\begin{array}{rcrcrcrcl} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 6 \end{array}$$
 and also 
$$\begin{array}{rcrcrcrcl} x & + & 2y & = & 3 \\ 4x & + & 5y & = & 9 \\ 7x & + & 8y & = & 15 \end{array}$$
. These lead to the

following augmented matrices:  $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 6 \\ 7 & 8 & : & 9 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 9 \\ 7 & 8 & : & 15 \end{bmatrix}$ . We can combine them together and get

$\begin{bmatrix} 1 & 2 & : & 3 & 3 \\ 4 & 5 & : & 6 & 9 \\ 7 & 8 & : & 9 & 15 \end{bmatrix}$  which we already know has the RREF of  $\begin{bmatrix} 1 & 0 & : & -1 & 1 \\ 0 & 1 & : & 2 & 1 \\ 0 & 0 & : & 0 & 0 \end{bmatrix}$  (from the last example —

only the :’s have moved). This corresponds to the augmented matrices  $\begin{bmatrix} 1 & 0 & : & -1 \\ 0 & 1 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 & : & 1 \\ 0 & 1 & : & 1 \\ 0 & 0 & : & 0 \end{bmatrix}$ .

These in turn tell us that the first system’s solution is  $x = -1$ ,  $y = 2$  and the second system’s solution is  $x = 1$  and  $y = 1$ .

This works because we aren’t mixing columns together (all operations are row operations). Also, notice that the same matrix can be interpreted in a number of ways. Before we had a single system in 3 variables and now we have 2 systems in 2 variables.

**Homework Problems:** For each of the following matrices perform Gauss-Jordan elimination (carefully following the algorithm defined above). I have given the result of the performing the forward pass to help you verify that you are on the right track. [Note: Each matrix has a unique RREF. However, REF is *not* unique. So if you do not follow my algorithm and instead either use the text’s method or your own random assortment of operations, you will almost certainly get different REFs along the way to the RREF.] Once you have completed row reduction, identify the pivots and pivot columns. Finally, interpret your matrices as a system or collection of systems of equations and note the corresponding solutions.

$$1. \quad \begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \Rightarrow \text{forward pass} \Rightarrow \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix}$$

$$2. \quad \begin{bmatrix} 2 & -4 & 0 & 4 \\ 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 4 & -8 & 1 & 6 \end{bmatrix} \Rightarrow \text{forward pass} \Rightarrow \begin{bmatrix} 2 & -4 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3. \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Rightarrow \text{forward pass} \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$4. \quad \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & -2 \\ 1 & -3 & -2 \end{bmatrix} \Rightarrow \text{forward pass} \Rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

It will be extremely useful to notice that when we perform elementary row operations on a matrix  $A$ , the linear relationships between columns of  $A$  do not change. Specifically... Suppose that  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are columns of  $A$ , and suppose that we perform some row operation and get  $A'$  whose corresponding columns are  $\mathbf{a}'$ ,  $\mathbf{b}'$ , and  $\mathbf{c}'$ . Then it turns out that:

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \quad \text{if and only if} \quad x\mathbf{a}' + y\mathbf{b}' + z\mathbf{c}' = \mathbf{0}$$

for some real numbers  $x$ ,  $y$ , and  $z$ .

This also holds for bigger or smaller collections of columns. Why? Essentially, since we are performing **row** operations, the relationships between **columns** are unaffected – we aren't "mixing" the columns together.

### Example:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow{2R1+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{-R2+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1 \times R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

In the RREF (on the far right) we have that the final column is 2 times the first column plus  $-1$  times the second column.

$$2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

So this must be true for all of the other matrices as well. In particular, we have that the third column of the original matrix is 2 times the first column plus  $-1$  times the second column:

$$2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

**Why is this important?** Well, first, Gauss-Jordan elimination typically requires a lot of computation. This correspondence gives you a way to check that you have row-reduced correctly! In some cases, the linear relations among columns are *obvious* and we can just *write down the RREF without performing Gaussian elimination at all!* We will see other applications of this correspondence later in the course.

**Example:** Let  $A$  be a  $3 \times 3$  matrix whose RREF is  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ . Suppose that we know the first column

of  $A$  is  $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$  and the second is  $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$ . Then, by the linear correspondence, we know that the third column

must be  $-1 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ . Therefore, the mystery matrix is  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ .

**Example:** Let  $B$  be a matrix whose RREF is  $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . Suppose that we know the first

**pivot** column (i.e. the second column) of  $B$  is  $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$  and the second pivot column (i.e. the fourth column)

is  $\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$ . Then, by the linear correspondence, we know that the third and fifth columns must be...

$$2 \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \quad \text{and} \quad 3 \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 5 \end{bmatrix}$$

Therefore, the mystery matrix is  $B = \begin{bmatrix} 0 & 2 & 4 & -1 & 8 \\ 0 & -1 & -2 & 4 & -11 \\ 0 & 3 & 6 & 2 & 5 \end{bmatrix}$ .

## Homework Problems:

5. Verify that the linear correspondence holds between the each matrix, its REF, and its RREF in the previous homework problems and examples.

6. I just finished row reducing the matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 5 & 0 \end{bmatrix}$  and got  $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ . Something's wrong. Just using the linear correspondence explain how I know something's wrong and then find the real RREF (without row reducing).

7. A certain matrix has the RREF of  $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . The first column of the matrix is  $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$  and the third column is  $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ . Find the matrix.

8. A certain matrix has the RREF of  $\begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ . The second column of the matrix is  $\begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$  and the last column is  $\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ . Find the matrix.

9. A certain matrix has the RREF of  $\begin{bmatrix} 1 & -5 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ . The first pivot column of the matrix is  $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ , the second pivot column is  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ , and the third pivot column is  $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ . Find the matrix.

10.  $2 \times 2$  matrices can row reduce to either  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , or  $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}$  for some  $c \in \mathbb{R}$ . What can be said about the original matrices in each of these cases?

## A Few Answers:

5. If you'd like a particular example worked out, just ask me.

6. According to the linear correspondence, the final column of my matrix should be 2 times the first pivot column minus the second pivot column. However,  $3 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 0-1 \\ 3-5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$  which does not match the third column of my original matrix. So the RREF must be wrong.

Let's find  $a$  and  $b$  so that  $a$  times column 1 plus  $b$  times column 2 is column 3. In the final column of the original matrix, the second entry is  $-1$ . So we must have  $b = -1$ . Thus  $a \cdot (\text{column 1}) =$

$(\text{column 2}) + (\text{column 3})$ . Adding those columns together, we get  $\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$ . This is 5 times the first

column. So  $a = 5$ . Therefore, the correct RREF is  $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ .

7. According to the RREF, the second column is 2 times the first column, the fourth column is obtained by subtracting the first column from the third column (i.e. the second pivot column), and the last

column is twice the first column plus 3 times the third column. This gives us  $\begin{bmatrix} 2 & 4 & -1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 1 & 8 \end{bmatrix}$ .

8. According to the RREF, the second column is twice the first column. Thus the first column must

be  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ . The last column is  $-2$  times the first column plus the second pivot column (i.e. column 3).

Thus the second pivot column must be  $\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 10 \\ 12 \end{bmatrix}$ . And finally, the fourth column is 5

times the first column plus 3 times the third column. So we get  $\begin{bmatrix} 1 & 2 & 6 & 23 & 4 \\ 2 & 4 & 8 & 34 & 4 \\ 3 & 6 & 10 & 45 & 4 \\ 4 & 8 & 12 & 56 & 4 \end{bmatrix}$ .

9. The second column is  $-5$  times the first, the fourth column is 3 times column 1 plus 2 times column 3 (the second pivot column), and the sixth column is  $-1$  times the first column, 4 times the third col-

umn, and 2 times the fifth column (the third pivot column). Thus we get  $\begin{bmatrix} 3 & -15 & 1 & 11 & 2 & 5 \\ -1 & 5 & -1 & -5 & 3 & 3 \\ 2 & -10 & 1 & 8 & 1 & 4 \end{bmatrix}$ .

10. If your RREF is the zero matrix, you must have started with the zero matrix (the result of doing or undoing any row operation on a zero matrix yields a zero matrix). If your RREF is the identity matrix (i.e.  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ), then the linear correspondence guarantees that both columns of the original matrix are non-zero and they must not be multiples of each other. If your RREF is the third option, then the first column of your original matrix was zeros and the second column was not a column made entirely of zeros. Finally, in the last case, the second column of the original matrix must be  $c$  times the first column (the second column is a multiple of the first column).