Jordan Form Examples

Examples of finding the Jordan form of a matrix.

Note: The "interface" command forces Maple to print arbitrarily large matrices instead of suppressing large matrix output.

```
> restart;
  with(LinearAlgebra):
  interface(rtablesize=infinity):
```

Example: Let's find a matrix which puts A (defined below) into Jordan form.

```
> A := <<2,-3,0>|<3,-4,0>|<2,-2,-1>>;
```

$$A := \begin{bmatrix} 2 & 3 & 2 \\ -3 & -4 & -2 \\ 0 & 0 & -1 \end{bmatrix} \quad (1)$$

```
> factor(CharacteristicPolynomial(A,t));
```

$$(t+1)^3 \quad (2)$$

Let's find the ranks of the first few $(A-(-1)I)^i$...

```
> Rank(IdentityMatrix(3));
  Rank((A-(-1)*IdentityMatrix(3))^1);
  Rank((A-(-1)*IdentityMatrix(3))^2);
  Rank((A-(-1)*IdentityMatrix(3))^3);
```

$$3$$
$$1$$
$$0$$
$$0 \quad (3)$$

So there are 3 - 1 = 2 (linearly independent) eigenvectors with eigenvalue -1. And there is 1 - 0 = 1 (linearly independent) generalized eigenvector (which is not a regular eigenvector) with eigenvalue -1. Thus we have a 2-chain and a 1-chain.

To find the 2-chain we need to find the generalized (non-regular) eigenvector. Any element in Ker $(A-(-1)I)^2$ which does not lie in Ker$(A-(-1)I)$ will do. So we will find a basis for Ker$(A-(-1)I)$ and complete it to a basis for Ker$(A-(-1)I)^2$. The extra vector that appears when we complete the basis will be the

```
> X := op(NullSpace((A-(-1)*IdentityMatrix(3))^1));
  Y := op(NullSpace((A-(-1)*IdentityMatrix(3))^2));

  F := <X[1]|X[2]|Y[1]|Y[2]|Y[3]>;
```

```
ReducedRowEchelonForm(F);
```

$$X := \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$F := \begin{bmatrix} -\frac{2}{3} & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{bmatrix} \quad \textbf{(4)}$$

Thus Y_1 lies in $\mathrm{Ker}(A-(-1)I)^2$ but not in $\mathrm{Ker}(A-(-1)I)$. So we set $q_1 = Y_1$ and then $q_2 = (A-(-1)I_3)\,Y_1$ to get our 2-chain.

```
> q[1] := Y[1];
  q[2] := (A-(-1)*IdentityMatrix(3)).Y[1];

  # end of the chain...
  ((A-(-1)*IdentityMatrix(3))^2).Y[1];
```

$$q_1 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 := \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(5)}$$

Now we need to find an element in $\mathrm{Ker}(A - (-1)I)$ which is independent of q_2.

```
> X := op(NullSpace((A-(-1)*IdentityMatrix(3))^1));
  F := <q[2]|X[1]|X[2]>;
  ReducedRowEchelonForm(F);
```

$$X := \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$F := \begin{bmatrix} 2 & -\frac{2}{3} & -1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(6)}$$

So X_1 does the trick. Letting $q_3 = X_1$ we have completed our basis of generalized eigenvectors.

```
> q[3] := X[1];
```

$$q_3 := \begin{bmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} \quad \textbf{(7)}$$

Putting q_1, q_2, q_3 together in a matrix Q, we can put A into Jordan form.

Note: I will reverse the order of q_1 and q_2 so that the 1's in the Jordan form appear above the diagonal. If we put the q's in order, the 1's will appear below the diagonal. By the way, some texts prefer to define Jordan forms with 1's below the diagonal.

```
> Q := <q[2]|q[1]|q[3]>;
  J := Q^(-1).A.Q;
```

$$Q := \begin{bmatrix} 2 & 0 & -\frac{2}{3} \\ -2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$J := \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \textbf{(8)}$$

<u>Example:</u> We will find the Jordan form (and a matrix to get A into Jordan form) where A is defined as follows:

<u>Note:</u> I created this matrix by conjugating my desired Jordan form by a "random" matrix consisting of integers between -3 and 3. If I had been more careful it wouldn't look so awful.

```
> A := Matrix(11, 11, [35389/9812, -16135/9812, -23543/4906,
  9937/9812, 3637/9812, -54211/9812, -40917/9812, 117133/9812,
  -36445/4906, 20087/9812, -12382/2453, 1189/4906, 4159/4906,
  -4242/2453, 1707/4906, -1905/4906, -2757/4906, -5405/4906,
  19797/4906, -5523/2453, 10797/4906, -5727/2453, -559/4906,
  -9271/4906, 6333/2453, -2911/4906, -5339/4906, -2929/4906,
  -2182/2453, 8027/2453, -6679/2453, -2815/4906, 3330/2453,
  405/2453, -7141/2453, -5021/2453, 9038/2453, -3956/2453,
  -7510/2453, -18167/4906, 46795/4906, -14637/2453, -5732/2453,
  72/2453, -7211/9812, 47789/9812, 29203/4906, -23123/9812,
  47257/9812, 41985/9812, 25093/9812, -93537/9812, 27979/4906,
  -28581/9812, 6984/2453, 11299/9812, -26013/9812, -26755/4906,
  18363/9812, 1459/9812, -30693/9812, -54359/9812, 148791/9812,
  -46213/4906, 19977/9812, -15088/2453, 4717/4906, -18581/4906,
  -12647/2453, 6739/4906, -6167/4906, -26749/4906, -3318/2453,
  29087/2453, -18907/2453, 10361/4906, -8357/2453, 3336/2453,
  -7235/2453, -12758/2453, 3273/2453, -1260/2453, -12891/2453,
  -20033/4906, 71559/4906, -19938/2453, 6079/2453, -11999/2453,
  15215/9812, -2561/9812, -17075/4906, 10443/9812, 227/9812,
  -30869/9812, -10623/9812, 29727/9812, 295/4906, 16545/9812,
  -7773/2453, -3193/9812, -3457/9812, 6071/4906, -11941/9812,
  -797/9812, 5939/9812, 6997/9812, 1615/9812, -2527/4906,
  25161/9812, 1875/2453, -14715/9812, 12521/9812, 19669/4906,
  -13579/9812, 5549/9812, 28381/9812, 15641/9812, -43481/9812,
  14473/4906, -23137/9812, 14064/2453]);
```

$$A := \left[\left[\frac{35389}{9812}, -\frac{16135}{9812}, -\frac{23543}{4906}, \frac{9937}{9812}, \frac{3637}{9812}, -\frac{54211}{9812}, -\frac{40917}{9812}, \frac{117133}{9812}, -\frac{36445}{4906}, \frac{20087}{9812}, -\frac{12382}{2453}\right], \quad \textbf{(9)}$$

$$\left[\frac{1189}{4906}, \frac{4159}{4906}, -\frac{4242}{2453}, \frac{1707}{4906}, -\frac{1905}{4906}, -\frac{2757}{4906}, -\frac{5405}{4906}, \frac{19797}{4906}, -\frac{5523}{2453}, \frac{10797}{4906}, -\frac{5727}{2453}\right],$$

$$\left[-\frac{559}{4906}, -\frac{9271}{4906}, \frac{6333}{2453}, -\frac{2911}{4906}, -\frac{5339}{4906}, -\frac{2929}{4906}, -\frac{2182}{2453}, \frac{8027}{2453}, -\frac{6679}{2453}, -\frac{2815}{4906}, \frac{3330}{2453}\right],$$

$$\left[\frac{405}{2453}, -\frac{7141}{2453}, -\frac{5021}{2453}, \frac{9038}{2453}, -\frac{3956}{2453}, -\frac{7510}{2453}, -\frac{18167}{4906}, \frac{46795}{4906}, -\frac{14637}{2453},\right.$$

$$-\frac{5732}{2453}, \frac{72}{2453}\Big],$$

$$\Big[-\frac{7211}{9812}, \frac{47789}{9812}, \frac{29203}{4906}, -\frac{23123}{9812}, \frac{47257}{9812}, \frac{41985}{9812}, \frac{25093}{9812}, -\frac{93537}{9812}, \frac{27979}{4906}, -\frac{28581}{9812}, \frac{6984}{2453}\Big],$$

$$\Big[\frac{11299}{9812}, -\frac{26013}{9812}, -\frac{26755}{4906}, \frac{18363}{9812}, \frac{1459}{9812}, -\frac{30693}{9812}, -\frac{54359}{9812}, \frac{148791}{9812}, -\frac{46213}{4906}, \frac{19977}{9812}, -\frac{15088}{2453}\Big],$$

$$\Big[\frac{4717}{4906}, -\frac{18581}{4906}, -\frac{12647}{2453}, \frac{6739}{4906}, -\frac{6167}{4906}, -\frac{26749}{4906}, -\frac{3318}{2453}, \frac{29087}{2453}, -\frac{18907}{2453}, \frac{10361}{4906}, -\frac{8357}{2453}\Big],$$

$$\Big[\frac{3336}{2453}, -\frac{7235}{2453}, -\frac{12758}{2453}, \frac{3273}{2453}, -\frac{1260}{2453}, -\frac{12891}{2453}, -\frac{20033}{4906}, \frac{71559}{4906}, -\frac{19938}{2453}, \frac{6079}{2453}, -\frac{11999}{2453}\Big],$$

$$\Big[\frac{15215}{9812}, -\frac{2561}{9812}, -\frac{17075}{4906}, \frac{10443}{9812}, \frac{227}{9812}, -\frac{30869}{9812}, -\frac{10623}{9812}, \frac{29727}{9812}, \frac{295}{4906}, \frac{16545}{9812}, -\frac{7773}{2453}\Big],$$

$$\Big[-\frac{3193}{9812}, -\frac{3457}{9812}, \frac{6071}{4906}, -\frac{11941}{9812}, -\frac{797}{9812}, \frac{5939}{9812}, \frac{6997}{9812}, \frac{1615}{9812}, -\frac{2527}{4906}, \frac{25161}{9812}, \frac{1875}{2453}\Big],$$

$$\Big[-\frac{14715}{9812}, \frac{12521}{9812}, \frac{19669}{4906}, -\frac{13579}{9812}, \frac{5549}{9812}, \frac{28381}{9812}, \frac{15641}{9812}, -\frac{43481}{9812}, \frac{14473}{4906}, -\frac{23137}{9812}, \frac{14064}{2453}\Big]\Big]$$

The LinearAlgebra package has a command "JordanForm" to do this for us. The option "output='Q'" returns the matrix which conjugates A into its Jordan form.

```
> JordanForm(A);
  Q := JordanForm(A,output='Q');
  Q^(-1).A.Q;
```

$$
\begin{bmatrix}
5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
$$

$$
Q := \Bigg[\Bigg[\frac{1751}{4906}, -\frac{11835225256960520826327}{13629225786822765262}, 0, -\frac{35205531204623}{1367022248922},
$$

$$
\frac{15154243871575163888650 3}{735978192488429324148}, -\frac{9651729611715029751}{1111229171367530 8}, -\frac{7363500110}{278642937},
$$

$$
\frac{3076803925008015091 3}{150015938134616658}, -\frac{8188915500390983933}{11112291713675308}, -\frac{53220704}{278642937},
$$

$$
\frac{25882204227108744723 1}{5100541896576966372}\Bigg],
$$

$$
\Bigg[\frac{1751}{9812}, -\frac{40410079022086851642187 1}{9267873535039480378 16}, 0, \frac{81457633615465}{2734044497844},
$$

$$
-\frac{1135401216437833913825735}{417054309076776617017 2}, -\frac{16484881430278565647 1}{377817918264960472}, \frac{8530585250}{278642937},
$$

$$
-\frac{462360838055622816457}{1700180632192322124}, -\frac{1398551789526249803 89}{377817918264960472}, \frac{129514387}{557285874},
$$

$$
-\frac{210721406287021390325}{3400361264384644248}\Bigg],
$$

$$
\Bigg[-\frac{1751}{4906}, \frac{1002135903222861760778 63}{115848419187993504727}, -\frac{17}{2453}, -\frac{2591717997947511994675 61}{12511629272303298510516},
$$

$$
\frac{2077617845607094932258 01}{104263577269194154254 3}, \frac{1634353547521497419 23}{188908959132480236}, -\frac{271575010307051257 75}{1275135474144241593},
$$

$$
\frac{338595215239270468517}{1700180632192322124}, \frac{1386581393745874412 57}{188908959132480236}, -\frac{21554366401852201 9}{1275135474144241593},
$$

$$
\frac{35404930260727919035}{784698753319533288}\Bigg],
$$

$$
\Bigg[-\frac{1751}{9812}, \frac{396792094902863823775005}{926787353503948037816}, -\frac{17}{2453}, -\frac{104223339510929811140011}{625581463615164925525 8},
$$

$$
\frac{2156843637441004085663 35}{139018103025592205672 4}, \frac{1617711706882650536 55}{377817918264960472}, -\frac{218166455280207043 15}{1275135474144241593},
$$

$$\frac{4395342102546369295}{283363438698720354}, \frac{137280717168711966277}{377817918264960472}, -\frac{325499961158675507}{2550270948288483186},$$
$$\frac{173742063742261708397}{5100541896576966372}\Bigg],$$
$$\Bigg[\frac{1751}{4906}, -\frac{200755150651572432111809}{231696838375987009454}, 0, -\frac{35186583484907}{1367022248922},$$
$$\frac{523730849753593011517 37}{245326064162809774716}, -\frac{163569335690716043555}{188908959132480236}, -\frac{7363500110}{278642937},$$
$$\frac{10645667452779818651}{50005312711538886}, -\frac{138799564469738632173}{188908959132480236}, -\frac{53220704}{278642937},$$
$$\frac{86383280129483648345}{1700180632192322124}\Bigg],$$
$$\Bigg[\frac{5253}{9812}, -\frac{110092064878426905186855}{842533957730861852 56}, \frac{17}{4906}, \frac{189944063140460145853 85}{625581463615164925 5258},$$
$$-\frac{153081572063336401615 09}{568710421468331750478}, -\frac{493777032004726818709}{377817918264960472}, \frac{80074093497396205 93}{2550270948288483186},$$
$$-\frac{340173042838569648 23}{1275135474144241593}, -\frac{418921921411896687639}{377817918264960472}, \frac{15994428323135647 7}{5100541896576966372},$$
$$-\frac{3537463386710401327}{425045158048080531}\Bigg],$$
$$\Bigg[-\frac{1751}{9812}, \frac{400859917046167386214983}{926787353503948037816}, 0, -\frac{30379423520857}{2734044497844},$$
$$\frac{1414182689055600537476305}{12511629272303298510516}, \frac{163309992495525355063}{377817918264960472}, -\frac{3178817072}{278642937},$$
$$\frac{577945174355193785483}{5100541896576966372}, \frac{138567948060668473333}{377817918264960472}, -\frac{47191159}{557285874},$$
$$\frac{231478257751116791767}{10201083793153932744}\Bigg],$$
$$\Bigg[\frac{1751}{9812}, -\frac{404100790220868516421871}{926787353503948037816}, 0, -\frac{30379423520857}{2734044497844},$$
$$\frac{1414009270631116920619657}{12511629272303298510516}, -\frac{164848814302785656471}{377817918264960472}, -\frac{3178817072}{278642937},$$
$$\frac{577945174355193785483}{5100541896576966372}, -\frac{139855178952624980389}{377817918264960472}, -\frac{47191159}{557285874},$$
$$\frac{231478257751116791767}{10201083793153932744}\Bigg],$$
$$\Bigg[\frac{1751}{9812}, -\frac{399062979151050970857859}{926787353503948037816}, 0, -\frac{4258096484527}{227837041487},$$
$$\frac{65350082938000465206614}{347545257563980514181}, -\frac{162791306105143978079}{377817918264960472}, -\frac{1783922726}{92880979},$$
$$\frac{26681254977507406429}{141681719349360177}, -\frac{138104715242528155653}{377817918264960472}, -\frac{13720538}{92880979},$$
$$\frac{112011806526058909 49}{283363438698720354}\Bigg],$$

$$\left[-\frac{1751}{9812}, \frac{401179585119562995301157}{926787353503948037816}, -\frac{17}{4906}, \frac{1269295353867864380 2607}{1251162927230329851 0516}, -\frac{293461092558065563 2995}{231696838375987009454}, \frac{16356071700877412 8783}{377817918264960472}, \frac{267430165562922232 7}{255027094828848318 6}, -\frac{2399775255866809037}{188908959132480236}, \frac{1387483306885064091 81}{377817918264960472}, \frac{51230450525380585}{510054189657696637 2}, -\frac{27557642818936045603}{10201083793153932744}\right],$$

$$\left[-\frac{1751}{9812}, \frac{400848805532122022407921}{926787353503948037816}, 0, \frac{3691664261931}{911348165948}, -\frac{19331295935262780508 4428}{31279073180758246276 29}, \frac{16356071700877412 8783}{377817918264960472}, \frac{389028380}{92880979}, -\frac{317239252682806188155}{5100541896576966372}, \frac{1387483306885064091 81}{377817918264960472}, \frac{7690993}{185761958}, -\frac{114184426650624993859}{10201083793153932744}\right]\right]$$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \textbf{(10)}$$

Let's go ahead and compute these things manually.

First, we should figure out what the eigenvalues of A are. This can be done directly using "Eigenvalues" or indirectly by computing "CharacteristicPolynomial" and factoring. Maple's characteristic polynomial is the same as that in Curtis: f(t) = det(tI-A) which differs from the Spence, Insel, Freidberg polynomial by a factor of $(-1)^n$ or in this case $(-1)^{11}=-1$.

```
> Eigenvalues(A);
```

$$\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad (11)$$

```
> -CharacteristicPolynomial(A,t);
```

$$-t^{11} + 34\,t^{10} - 514\,t^{9} + 4560\,t^{8} - 26385\,t^{7} + 104622\,t^{6} - 290388\,t^{5} + 564888\,t^{4} - 755760\,t^{3} + 663200\,t^{2} - 344000\,t + 80000 \quad (12)$$

```
> -factor(CharacteristicPolynomial(A,t));
```

$$-(t-5)^{4}\,(t-2)^{7} \quad (13)$$

Maple can compute the minimal polynomial of a matrix. Notice that A satisfies both of the characteristic polynomial (by the Cayley-Hamilton Theorem) and minimal polynomial (by definition).

```
> factor(MinimalPolynomial(A,t));
```

$$(t-5)^{2}\,(t-2)^{3} \quad (14)$$

```
> simplify(subs(t=A,CharacteristicPolynomial(A,t)));

  simplify(subs(t=A,MinimalPolynomial(A,t)));
```

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \quad \textbf{(15)}$$

So far we know that A has two eigenvalues: $\lambda = 5$ with algebraic multiplicity 4 and $\lambda = 2$ with algebraic multiplicity 7. Thus we need to find 4 (linearly independent) generalized eigenvectors with eigenvalue 5 and 7 with eigenvalue 2. But more than that, these vectors need to parts of chains if we are to get our matrix into Jordan form.

To figure out the length of the chains we need to compute ranks of $\left(A - \lambda I\right)^i$ for $i = 0 \ldots 11$ (Actually we only need to compute ranks up to the point we get a repeat. This will happen by the time we hit the eigenvalue's algebraic multiplicity. On the other hand, Maple can happily compute more ranks than we really need).

```
> seq(Rank((A-5*IdentityMatrix(11))^i),i=0..11);
```

$$11, 8, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7 \quad \textbf{(16)}$$

```
> seq(Rank((A-2*IdentityMatrix(11))^i),i=0..11);
```

$$11, 8, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4 \quad \textbf{(17)}$$

So we have that the nullity of $A - 5\,I$ is $11 - 8 = 3$ and the nullity of $(A - 5\,I)^2$ is $11 - 7 = 4$ (as is the nullity of $(A - 5\,I)^i$ for all $i \geq 2$). This means we have 3 linearly independent eigenvectors with eigenvalue 5 and 1 more generalized eigenvector (which is not a regular eigenvector). Thus we should be able to find 1 chain of length 2 and 2 chains of length 1.

The nullity of $A - 2\,I$ is $11 - 8 = 3$, the nullity of $(A - 2\,I)^2$ is $11 - 5 = 6$, and the nullity of $(A - 2\,I)^3$ is $11 - 4 = 7$ (as is the nullity of $(A - 2\,I)^i$ for all $i \geq 3$). This means we have 3 linearly independent eigenvectors with eigenvalue 2, $6 - 3 = 3$ more generalized eigenvectors in Ker $(A - 2\,I)^2$ which are not regular eigenvectors, and $7 - 6 = 1$ generalized eigenvector in Ker $(A - 2\,I)^3$ which are not Ker $(A - 2\,I)^2$. Thus we will have 1 chain of length 3 and 2 chains of length 2.

Let's find these chains.

First, $\lambda = 5$. We will start with the chain of length 2. This is generated by the non-regular generlized eigenvector. We need something in Ker $(A-5\,I)^2$ which isn't in Ker $(A-2\,I)$. So we'll find a basis for Ker $(A-2\,I)$ and complete it to a basis for Ker $(A-5\,I)^2$. The new vector that shows up when we complete the basis will be the vector we're looking for.

```
> X := op(NullSpace((A-5*IdentityMatrix(11))^1));
  Y := op(NullSpace((A-5*IdentityMatrix(11))^2));

  F := <X[1]|X[2]|X[3]|Y[1]|Y[2]|Y[3]|Y[4]>;
  ReducedRowEchelonForm(F);
```

$$X := \begin{bmatrix} 2 \\ 0 \\ -1 \\ -4 \\ 6 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -4 \\ 5 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} \frac{5}{2} \\ 0 \\ -2 \\ -\frac{7}{2} \\ \frac{15}{2} \\ \frac{5}{2} \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -4 \\ 5 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F := \begin{bmatrix} 2 & 2 & 2 & \frac{5}{2} & -\frac{1}{2} & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & -2 & -1 & -2 & 1 & -2 & -1 \\ -4 & -4 & -1 & -\frac{7}{2} & -\frac{1}{2} & -4 & -1 \\ 6 & 5 & 3 & \frac{15}{2} & -\frac{3}{2} & 5 & 3 \\ 1 & 1 & 3 & \frac{5}{2} & -\frac{3}{2} & 1 & 3 \\ -2 & -2 & -1 & -3 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(18)}$$

So the first vector in the basis for Ker $(A-5\,I)^2$ completes the basis from Ker $(A-5\,I)$. Thus Y_1 is a generlized eigenvector which generates our chain of length 2. Let's call the vectors in this chain q_1 and q_2.

```
> q[1] := Y[1];
  q[2] := (A-5*IdentityMatrix(11)).Y[1];

  # end of the chain...
  ((A-5*IdentityMatrix(11))^2).Y[1];
```

$$q_1 := \begin{bmatrix} \frac{5}{2} \\ 0 \\ -2 \\ -\frac{7}{2} \\ \frac{15}{2} \\ \frac{5}{2} \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_2 := \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ -1 \\ -\frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

Next, we need to find our 1 chains (eigenvectors) which aren't among those we've already found (i.e. q_2). So we find a basis for $\mathrm{Ker}(A - 5\,I)$ and find which of these vectors completes the basis starting with the eigenvector we already know.

```
> X := op(NullSpace((A-5*IdentityMatrix(11))^1));
  F := <q[2]|X[1]|X[2]|X[3]>;
  ReducedRowEchelonForm(F);
```

$$X := \begin{bmatrix} 2 \\ 0 \\ -1 \\ -4 \\ 6 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -4 \\ 5 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ 3 \\ 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F := \begin{bmatrix} -1 & 2 & 2 & 2 \\ -\frac{1}{2} & 0 & 0 & 1 \\ 1 & -1 & -2 & -1 \\ \frac{1}{2} & -4 & -4 & -1 \\ -1 & 6 & 5 & 3 \\ -\frac{3}{2} & 1 & 1 & 3 \\ \frac{1}{2} & -2 & -2 & -1 \\ -\frac{1}{2} & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \textbf{(20)}$$

The first two vectors in the basis for Ker$(A - 5\,I)$ do the job. We'll call these q_3 and q_4.

```
> q[3] := X[1];
  q[4] := X[2];
```

$$q_3 := \begin{bmatrix} 2 \\ 0 \\ -1 \\ -4 \\ 6 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$q_4 := \begin{bmatrix} 2 \\ 0 \\ -2 \\ -4 \\ 5 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(21)}$$

Now we move onto $\lambda = 2$.

First, let's find our chain of length 3. We need a vector in $\mathrm{Ker}(A - 2\,I)^3$ which does not belong to $\mathrm{Ker}(A - 2)^2$. So again we take basis for $\mathrm{Ker}(A - 2\,I)^2$ and complete it to a basis for $\mathrm{Ker}(A - 2)^3$. The extra vector that shows up will be the one we're looking for.

```
> X := op(NullSpace((A-2*IdentityMatrix(11))^2));
  Y := op(NullSpace((A-2*IdentityMatrix(11))^3));

  F := <X[1]|X[2]|X[3]|X[4]|X[5]|X[6]|Y[1]|Y[2]|Y[3]|Y[4]|Y[5]|Y[6]|Y[7]>;
  ReducedRowEchelonForm(F);
```

$$X := \left[\begin{array}{c} \frac{25}{2} \\ -\frac{23}{6} \\ \frac{29}{6} \\ \frac{5}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 20 \\ -\frac{25}{3} \\ \frac{28}{3} \\ \frac{11}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -15 \\ \frac{20}{3} \\ -\frac{20}{3} \\ -\frac{7}{3} \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -5 \\ \frac{4}{3} \\ -\frac{7}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$$Y := \left[\begin{array}{c} \frac{25}{2} \\ -\frac{23}{6} \\ \frac{29}{6} \\ \frac{5}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 20 \\ -\frac{25}{3} \\ \frac{28}{3} \\ \frac{11}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -\frac{245}{8} \\ \frac{101}{8} \\ -\frac{115}{8} \\ -\frac{25}{4} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{125}{8} \\ -\frac{143}{24} \\ \frac{185}{24} \\ \frac{47}{12} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -5 \\ \frac{4}{3} \\ -\frac{7}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

$F :=$

$$\left[\left[\frac{25}{2}, \frac{1}{2}, 20, -15, \frac{1}{2}, -5, \frac{25}{2}, \frac{1}{2}, 20, -\frac{245}{8}, \frac{125}{8}, \frac{1}{2}, -5\right],\right.$$

$$\left[-\frac{23}{6}, -\frac{1}{2}, -\frac{25}{3}, \frac{20}{3}, -\frac{1}{2}, \frac{4}{3}, -\frac{23}{6}, -\frac{1}{2}, -\frac{25}{3}, \frac{101}{8}, -\frac{143}{24}, -\frac{1}{2}, \frac{4}{3}\right],$$

$$\left[\frac{29}{6}, \frac{3}{2}, \frac{28}{3}, -\frac{20}{3}, -\frac{1}{2}, -\frac{7}{3}, \frac{29}{6}, \frac{3}{2}, \frac{28}{3}, -\frac{115}{8}, \frac{185}{24}, -\frac{1}{2}, -\frac{7}{3}\right],$$

$$\left[\frac{5}{3}, 2, \frac{11}{3}, -\frac{7}{3}, 0, -\frac{2}{3}, \frac{5}{3}, 2, \frac{11}{3}, -\frac{25}{4}, \frac{47}{12}, 0, -\frac{2}{3}\right],$$

$$\left[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1\right],$$

$$\left[0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0\right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0\right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0\right],$$

$$\left[0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0\right],$$

$$\left[0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0\right],$$

$$\left[1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\right]\Big]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(22)}$$

This time Y_4 gives us the missing basis element. Let's call this vector q_5 and generate its chain (calling the other two vectors q_6 and q_7).

```
> q[5] := Y[4];
  q[6] := (A-2*IdentityMatrix(11)).Y[4];
  q[7] := ((A-2*IdentityMatrix(11))^2).Y[4];

  # end of the chain...
  ((A-2*IdentityMatrix(11))^3).Y[1];
```

$$q_5 := \begin{bmatrix} -\frac{245}{8} \\ \frac{101}{8} \\ -\frac{115}{8} \\ -\frac{25}{4} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$q_6 := \begin{bmatrix} \frac{37}{8} \\ \frac{19}{4} \\ -\frac{87}{4} \\ -\frac{107}{8} \\ \frac{29}{8} \\ \frac{105}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ -\frac{35}{8} \\ -\frac{9}{2} \\ \frac{69}{8} \end{bmatrix}$$

$$q_7 := \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ -\frac{1}{4} \\ 0 \\ 0 \\ 0 \\ \frac{1}{4} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

Now let's find our two chains of length 2. We need to find vectors in $\mathrm{Ker}(A - 2I)^2$ which don't lie in $\mathrm{Ker}(A - 2I)$ and also don't appear in our previously computed chain. So we'll find a basis for Ker $(A - 2I)$ complete it to a basis for $\mathrm{Ker}(A - 2I)^2$ chucking q_6 in front to rule it out. (Why q_6? Because it lives at the same "level" as the beginning of our two new chains.)

```
> X := op(NullSpace((A-2*IdentityMatrix(11))^1));
  Y := op(NullSpace((A-2*IdentityMatrix(11))^2));

  F := <q[6]|X[1]|X[2]|X[3]|Y[1]|Y[2]|Y[3]|Y[4]|Y[5]|Y[6]>;
  ReducedRowEchelonForm(F);
```

$$X := \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ -1 \\ 0 \\ \frac{4}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ -\frac{5}{3} \\ 1 \\ 1 \\ \frac{5}{3} \\ 0 \\ \frac{2}{3} \\ \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Y := \begin{bmatrix} \frac{25}{2} \\ -\frac{23}{6} \\ \frac{29}{6} \\ \frac{5}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ -\frac{25}{3} \\ \frac{28}{3} \\ \frac{11}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -15 \\ \frac{20}{3} \\ -\frac{20}{3} \\ -\frac{7}{3} \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ \frac{4}{3} \\ -\frac{7}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F := \begin{bmatrix} \frac{37}{8} & \frac{4}{3} & 0 & \frac{5}{3} & \frac{25}{2} & \frac{1}{2} & 20 & -15 & \frac{1}{2} & -5 \\ \frac{19}{4} & -\frac{1}{3} & 0 & -\frac{5}{3} & -\frac{23}{6} & -\frac{1}{2} & -\frac{25}{3} & \frac{20}{3} & -\frac{1}{2} & \frac{4}{3} \\ -\frac{87}{4} & -1 & 2 & 1 & \frac{29}{6} & \frac{3}{2} & \frac{28}{3} & -\frac{20}{3} & -\frac{1}{2} & -\frac{7}{3} \\ -\frac{107}{8} & 0 & 2 & 1 & \frac{5}{3} & 2 & \frac{11}{3} & -\frac{7}{3} & 0 & -\frac{2}{3} \\ \frac{29}{8} & \frac{4}{3} & 0 & \frac{5}{3} & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{105}{8} & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{8} & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{8} & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{35}{8} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{9}{2} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{69}{8} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{5} & \frac{16}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{53}{10} & -\frac{131}{5} & 0 & \frac{73}{10} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{26}{5} & \frac{79}{5} & -1 & -\frac{16}{5} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{5}{2} & 14 & 0 & -\frac{7}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{8}{5} & -\frac{7}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{8}{5} & -\frac{7}{5} & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(24)}$$

The first two vectors in the new basis do the trick: Y_1 and Y_2. Let's call them q_8 and q_{10} (calling the second element in these chains q_9 and q_{11} respectively).

```
> q[8]  := Y[1];
  q[9]  := (A-2*IdentityMatrix(11)).Y[1];

  q[10]  := Y[2];
  q[11]  := (A-2*IdentityMatrix(11)).Y[2];
```

$$q_8 := \begin{bmatrix} \frac{25}{2} \\ -\frac{23}{6} \\ \frac{29}{6} \\ \frac{5}{3} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$q_9 := \begin{bmatrix} -\frac{1}{6} \\ -\frac{8}{3} \\ 9 \\ \frac{37}{6} \\ -\frac{1}{6} \\ -\frac{29}{6} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{13}{6} \\ 2 \\ -\frac{17}{6} \end{bmatrix}$$

$$q_{10} := \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_{11} := \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \quad \textbf{(25)}$$

Now we have a basis made entirely of chains of generalized eigenvectors. I will reverse the order of each of these chains so our "1's" in our Jordan form appear above (instead of below) the main diagonal.

```
> P := <q[2]|q[1]|q[3]|q[4]|q[7]|q[6]|q[5]|q[9]|q[8]|q[11]|q[10]>;
```

$$P := \begin{bmatrix} -1 & \frac{5}{2} & 2 & 2 & 0 & \frac{37}{8} & -\frac{245}{8} & -\frac{1}{6} & \frac{25}{2} & -\frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{19}{4} & \frac{101}{8} & -\frac{8}{3} & -\frac{23}{6} & 1 & -\frac{1}{2} \\ 1 & -2 & -1 & -2 & \frac{1}{2} & -\frac{87}{4} & -\frac{115}{8} & 9 & \frac{29}{6} & 0 & \frac{3}{2} \\ \frac{1}{2} & -\frac{7}{2} & -4 & -4 & \frac{1}{2} & -\frac{107}{8} & -\frac{25}{4} & \frac{37}{6} & \frac{5}{3} & -\frac{1}{2} & 2 \\ -1 & \frac{15}{2} & 6 & 5 & 0 & \frac{29}{8} & 0 & -\frac{1}{6} & 0 & -\frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & 1 & 1 & -\frac{1}{4} & \frac{105}{8} & 0 & -\frac{29}{6} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & -3 & -2 & -2 & 0 & \frac{1}{8} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{8} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & 1 & 0 & -\frac{35}{8} & 0 & \frac{13}{6} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 & 0 & \frac{1}{4} & -\frac{9}{2} & 0 & 2 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & \frac{69}{8} & 0 & -\frac{17}{6} & 1 & -\frac{1}{2} & 0 \end{bmatrix} \qquad (26)$$

```
> P^(-1).A.P;
```

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad (27)$$