Math 4710/5710 # Separation Axioms **Fall 2014**

We now come to the seedy underbelly of topology. It is hard to believe in this day and age that there are those who still practice radical segregation – even advocate for it! In the time of Felix Hausdorff (1868–1942) and Pavel Urysohn (1898–1924) this was accepted culturally, but how such separation persists today is hard to understand. You see, topologists love to separate. They have even convinced themselves that it's a "regular" or "normal" thing.

Before we can begin to battle these prejudices, we must better understand them. Let's explore this despicable list of axioms in order that we may effect a change and move towards a world which isn't T_4 or T_3. Maybe our children can grow up not knowing the horrors of Hausdorff "spaces" and never having been inflicted with Urysohn's innocent sounding "lemma". Stand up for what's right! Be the change you believe in!

Let X be a topological space. We say two points (or more generally two sets) are **topologically distinguishable** if there is an open set containing one but not the other.

Two sets are **separated** if each set is disjoint from the other set's closure. So $A, B \subseteq X$ are **separated** if $A \cap \overline{B} = \varnothing$ and $\overline{A} \cap B = \varnothing$. [Note: This does not mean that their closures are disjoint!] Notice that separated objects are topologically distinguishable (B is disjoint from $X - \overline{B}$ and $X - \overline{B}$ is an open set containing A).

Points (or more generally sets) are **separated by open sets** if the points (or sets) are contained in disjoint open sets. That is, $A, B \subseteq X$ are **separated by open sets** if there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. As before notice that "separated by open sets" implies plain old "separated".

Sets are **separated by continuous functions** if there is a continuous function into the reals which takes on the value of 0 at every point in the first set and 1 at every point in the second set. That is, $A, B \subseteq X$ are **separated by a continuous function** $f : X \to [0, 1]$ if f is continuous, $f(a) = 0$ for all $a \in A$, and $f(b) = 1$ for all $b \in B$. Notice "separated by a continuous function" implies "separated by open sets" since $A \subseteq U = f^{-1}(-1/2, 1/2)$ and $B \subseteq V = f^{-1}(1/2, 3/2)$ where U and V are open (since f is continuous) and disjoint (because $(-1/2, 1/2)$ and $(1/2, 3/2)$ are disjoint).

Hausdorff A topological space is called **Hausdorff** if points can be separated by open sets. So for all $x, y \in X$ such that $x \neq y$, there exists open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Regular A topological space is called **regular** if points and closed sets can be separated by open sets. Specifically, for all $x \in X$ and all closed sets $A \subseteq X$ with $x \notin A$, there exists open sets $U, V \subseteq X$ such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.

Completely Regular A topological space is called **completely regular** if points and closed sets can be separated by continuous functions. Specifically, for all $x \in X$ and all closed sets $A \subseteq X$ with $x \notin A$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$.

Normal A topological space is called **normal** if closed sets can be separated by open sets. Specifically, for all closed sets $A, B \subseteq X$ with $A \cap B = \varnothing$, there exists open sets $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$.

Note: Urysohn's Lemma (see below) tells us that we don't need to define a collection of "Completely Normal" spaces. If we did, we would find that completely normal and normal define the same collection of spaces.

$\mathbf{T}_0$ A topological space is said to be $\mathbf{T}_0$ if all points are topologically distinguishable. This means that given $x, y \in X$ such that $x \neq y$ there is an open set $U \subseteq X$ such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

$\mathbf{T}_1$ A topological space is said to be $\mathbf{T}_1$ if all points are separated. This means that there exists open sets $U, V \subseteq X$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$. This is equivalent to saying $x \notin \overline{\{y\}}$ and $y \notin \overline{\{x\}}$. In other words, being $\mathbf{T}_1$ is equivalent to $\overline{\{x\}} = \{x\}$ for all $x \in X$ – that is – *singletons are closed.*

$\mathbf{T}_2$ A topological space is said to be $\mathbf{T}_2$ if it's Hausdorff: $\mathbf{T}_2 =$ Hausdorff.

$\mathbf{T}_3$ A topological space is said to be $\mathbf{T}_3$ if it's regular and $\mathbf{T}_0$: $\mathbf{T}_3 =$ regular + $\mathbf{T}_0$.

$\mathbf{T}_{3^1/_2}$ A topological space is said to be $\mathbf{T}_{3^1/_2}$ if it's completely regular and $\mathbf{T}_0$: $\mathbf{T}_{3^1/_2} =$ completely regular + $\mathbf{T}_0$.

$\mathbf{T}_4$ A topological space is said to be $\mathbf{T}_4$ if it's normal and $\mathbf{T}_1$: $\mathbf{T}_4 =$ normal + $\mathbf{T}_1$.

It can be shown that $\mathbf{T}_4 \Rightarrow \mathbf{T}_{3^1/_2} \Rightarrow \mathbf{T}_3 \Rightarrow \mathbf{T}_2 \Rightarrow \mathbf{T}_1 \Rightarrow \mathbf{T}_0$. There are examples which show none of the implications can be reversed. Also, please be careful. What Munkres calls normal, completely regular, and regular are more acurately $\mathbf{T}_4$, $\mathbf{T}_{3^1/_2}$, and $\mathbf{T}_3$.

Urysohn's Lemma: [Normal $\Longrightarrow$ Completely Normal] In a normal space, disjoint closed sets can be separated by a continuous function. Specifically, let X be normal with $A, B \subseteq X$ such that A and B are closed and $A \cap B = \varnothing$. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

First Countable A topological space X is **first countable** if at each $x \in X$ there is a countable neighborhood basis. This means there is a countable collection $\mathcal{B}$ of neighborhoods of x such that for every neighborhood U of x there exists some $B \in \mathcal{B}$ such that $B \subseteq U$. [Every metric space is first countable: Consider balls centered at $x \in X$ with positive rational radii.]

Second Countable A topological space X is **second countable** if it has a countable basis. [$\mathbb{R}^n$ is second countable: Consider balls of rational radii centered at points with rational coordinates.]

Separable We say D is **dense** in X if $\overline{D} = X$. X is said to be **separable** if it contains a countable dense set. [For example: $\overline{\mathbb{Q}} = \mathbb{R}$ so $\mathbb{R}$ is separable. More generally, any second countable space is separable.]

Lindelöf If every open cover has a finite subcover, our space is **compact**. If every *countable* open cover (an open cover with only countably many elements) has a finite subcover, then our space is **countably compact**. If every open cover has a *countable* subcover, our space is **Lindelöf**. [Obviously, Lindelöf + countably compact = compact. Also, every second countable space is Lindelöf.]

Theorem: $\mathbf{T}_3$ + second countable $\Rightarrow$ $\mathbf{T}_4$.

Theorem: metrizable $\Rightarrow$ $\mathbf{T}_4$.

Theorem: compact + Hausdorff $\Rightarrow$ $\mathbf{T}_4$.

Theorem: Well ordered sets with order topology are $\mathbf{T}_4$.

Urysohn's Metrization Theorem: $\mathbf{T}_3$ + second countable $\Rightarrow$ metrizable.

An Embedding Theorem: A space is $\mathbf{T}_{3^1/_2}$ if and only if it is homeomorphic to a subspace of $[0,1]^J$ for some J.

Locally Finite Let X be a topological space and $\mathcal{A} \subseteq \mathcal{P}(X)$. We say that $\mathcal{A}$ is **locally finite** in X if for every $x \in X$ there exists some neighborhood U of x such that U is disjoint from all but finitely many elements of $\mathcal{A}$. Similarly, $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be **countably locally finite** (or **σ-locally finite**) in X if $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is locally finite in X. This means $\mathcal{B}$ is a countable union of locally finite sets.

Open Refinement Let $\mathcal{A} \subseteq \mathcal{P}(X)$. We say that $\mathcal{B}$ is an **refinement** of $\mathcal{A}$ if for every $A \in \mathcal{A}$ there is an element $B \in \mathcal{B}$ such that $B \subseteq A$. We say that $\mathcal{B}$ is an **open refinement** of $\mathcal{A}$ if $\mathcal{B}$ is a refinement of $\mathcal{A}$ and every element of $\mathcal{B}$ is an open set.

Paracompact A topological space X is **paracompact** if every open cover of X has a locally finite open refinement which covers X.

Locally Metrizable X is **locally metrizable** if every $x \in X$ has a metrizable neighborhood.

Nagata-Smirnov Metrization Theorem: A space is metrizable if and only if it is $\mathbf{T}_3$ and has a countably locally finite basis.

Theorem: Hausdorff + paracompact $\Rightarrow$ $\mathbf{T}_4$.

Theorem: $\mathbf{T}_3$ + Lindelöf $\Rightarrow$ paracompact.

Smirnov Metrization Theorem: metrizable $\Leftrightarrow$ Hausdorff + paracompact + locally metrizable.

Baire Space X is a **Baire space** if given any countable collection $\mathcal{A} = \{A_n \mid n = 1, 2, \dots\}$ of closed sets of X such that $\text{int}(A_n) = \varnothing$, we have $\text{int}\left(\bigcup_{n=1}^{\infty} A_n\right) = \varnothing$. [Recall that $\text{int}(A)$ is the *interior* of A. This is the union of all open sets contained in A. So $\text{int}(A) = \varnothing$ if A contains no non-empty open sets.] In Baire's terminology, a set A is of **first category** if it is contained in the union of a countable collection of closed sets having empty interiors. All other sets are of **second category**. So a Baire space is a space whose non-empty open sets are all of the second category.

Baire Category Theorem: compact Hausdorff spaces are Baire spaces.
Also, complete metric spaces are Baire spaces.