Math 2110 **Test #2** **November 2^{nd}, 2020**

Name: ______________________________ **Be sure to show your work!**

1. (20 points) Divides and Induction

(a) Give a careful definition of the set of odd integers using *set builder* notation.

$\mathbb{O} =$

(b) Suppose the $a, b, c \in \mathbb{Z}$ and that a divides b and a divides c. Prove that a divides $b + c$.

(c) Using induction, prove that for any positive integer n, we have $\displaystyle\sum_{i=1}^{n}(2i - 1) = n^2$.

2. (20 points) Contradiction and Strategy

(a) Suppose I want to prove the statement: "Given any $m \in \mathbb{Z}$, there exists a unique $n \in \mathbb{Z}$ such that $m < \frac{n}{2} < m + 1$."
In addition, suppose we have already shown existence. Sketch the beginning and end of the next part of the proof.

(b) Prove there is no smallest integer.

(c) Prove $\sqrt{2}$ is irrational.

3. (20 points) Sequences

(a) Give the definition of $\langle a_n \rangle_{n=0}^{\infty} \to L$.

(b) Show $\langle n^2 + 1 \rangle_{n=0}^{\infty}$ is bounded below but not above.

(c) Prove that $\left\langle \dfrac{n+5}{3n^2+9n-1} \right\rangle_{n=1}^{\infty} \to 0$.

The following calculation might be helpful: $\dfrac{n+5}{3n^2+9n-1} \leq \dfrac{n+5n}{3n^2+9n-1} < \dfrac{6n}{3n^2-1} \leq \dfrac{6n}{3n^2-n^2} = \dfrac{6n}{2n^2} = \dfrac{3}{n}$

4. (20 points) Sets!

(a) Let $\mathcal{A}$ be a set of subsets of $\mathbb{R}$ (i.e., $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$). Define what "$x \in \bigcup \mathcal{A}$" means.

(b) Let $A = \{12n \mid n \in \mathbb{Z}\}$ and $B = \{x \mid x \in \mathbb{Z}$ and x is divisible by $20\}$.
Also, let $C = \{n \mid$ there exists some $k \in \mathbb{Z}$ such that $n = 2k\}$. Prove that $A \cup B \subseteq C$.

(c) Let $f : A \to B$ and $T_1, T_2 \subseteq B$. Prove that $f^{-1}(T_1 \cap T_2) = f^{-1}(T_1) \cap f^{-1}(T_2)$.

5. (20 points) Functions!

(a) Let $f : A \to B$. Define what it means for f to be invertible.

(b) Suppose $f : A \to B$ and $g : B \to C$ are onto. Show that $g \circ f : A \to C$ is onto.

(c) Consider $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(n) = 5n - 2$. Prove that f is one-to-one but not onto.