Math 2510 **Test #2** **April 2^{nd}, 2010**

Name: Answer Key

Don't merely state answers, prove your statements. **Be sure to show your work!**

1. (28 points) Converging Questions

(a) Prove that $\left\langle \dfrac{3n^2}{n^2+2n+3} \right\rangle$ converges.

To come up with a proof first we'll need to do some "scratch work". Using a little Calculus I insight, we see that this sequence's limit is 3. Thus we'll need $\left| \dfrac{3n^2}{n^2+2n+3} - 3 \right| < \epsilon$. So we should find *overestimates* of the fraction until we have something simple enough to set equal to ϵ and solve for n.

$$\left| \frac{3n^2}{n^2+2n+3} - 3 \right| = \left| \frac{3n^2 - 3(n^2+2n+3)}{n^2+2n+3} \right| = \left| \frac{-6n-9}{n^2+2n+3} \right| = \frac{6n+9}{n^2+2n+3} \leq \frac{6n+9n}{n^2+2n+3} < \frac{15n}{n^2} = \frac{15}{n}$$

The "$\leq$" following because increasing the numerator increases the fraction. Likewise, the "$<$" holds because decreasing the denominator increases the fraction.

Now we have a simple estimate for our fraction: $\dfrac{15}{n}$. So $15/n = \epsilon \Longrightarrow 15/\epsilon = n$. Now we're ready to write our proof.

Proof:

Suppose $\epsilon > 0$. Let $N = \left\lceil \dfrac{15}{\epsilon} \right\rceil$ ["$\lceil x \rceil$" is the ceiling function which rounds x up to the next integer].

Suppose $n \geq N$, we have that $n \geq \left\lceil \dfrac{15}{\epsilon} \right\rceil \geq \dfrac{15}{\epsilon}$. Therefore, $\dfrac{15}{\epsilon} \leq n$ and so $\dfrac{15}{n} \leq \epsilon$.

$$\left| \frac{3n^2}{n^2+2n+3} - 3 \right| = \left| \frac{3n^2 - 3(n^2+2n+3)}{n^2+2n+3} \right| = \left| \frac{-6n-9}{n^2+2n+3} \right| = \frac{6n+9}{n^2+2n+3} \leq \frac{6n+9n}{n^2+2n+3} < \frac{15n}{n^2} = \frac{15}{n}$$

Therefore, $\left| \dfrac{3n^2}{n^2+2n+3} - 3 \right| < \dfrac{15}{n} \leq \epsilon$ for all $n \geq N$. So our sequence converges to 3.

(b) Show that $\left\langle \dfrac{\sin(n)}{n^4+1} \right\rangle$ converges.

Hint: Ignore $\sin(n)$, then prove $\sin(n)$ is bounded and use a theorem.

Note: The hint isn't that helpful. I originally intended everyone to notice that $-1 \leq \sin(n) \leq 1$ and thus $|\sin(n)| \leq 1$ so that $\sin(n)$ is bounded and then prove that $1/(n^4+1)$ converges to 0. Once these facts are established, we simply quote the theorem that says "a bounded sequence $\times$ a sequence which converges to zero is a sequence which converges to zero". However, it's no harder to prove that $\sin(n)/(n^4+1)$ converges to 0 than proving $1/(n^4+1)$ converges to 0. So we'll just proceed directly.

Same as before, we need to do some scratchwork first.

$$\left| \frac{\sin(n)}{n^4+1} - 0 \right| = \frac{|\sin(n)|}{n^4+1} \leq \frac{1}{n^4+1} < \frac{1}{n^4}$$

So if $1/n^4 = \epsilon$, then $n^4 = 1/\epsilon$ and thus $n = \sqrt[4]{1/\epsilon}$. Now we're ready to write our proof.

Proof:

Suppose $\epsilon > 0$. Let $N = \left\lceil \sqrt[4]{\dfrac{1}{\epsilon}} \right\rceil$ ["$\lceil x \rceil$" is the ceiling function which rounds x up to the next integer].

Suppose $n \geq N$. $n \geq \left\lceil \sqrt[4]{\frac{1}{\epsilon}} \right\rceil \geq \sqrt[4]{\frac{1}{\epsilon}} \Rightarrow n^4 \geq \frac{1}{\epsilon} \Rightarrow \frac{1}{n^4} \leq \epsilon \Rightarrow \left| \frac{\sin(n)}{n^4+1} - 0 \right| = \frac{|\sin(n)|}{n^4+1} \leq \frac{1}{n^4+1} < \frac{1}{n^4}$.

Note that the "$\leq$" holds because the numerator "$|\sin(n)|$" is never larger than 1 because $-1 \leq \sin(n) \leq 1$. The "$<$" holds because $n^4+1 > n^4$ and decreasing a denominator increases a fraction.

Therefore, $\left| \frac{\sin(n)}{n^4+1} - 0 \right| < \frac{1}{n^4} \leq \epsilon$ for all $n \geq N$. So our sequence converges to 0.

(c) Prove that $\left\langle \frac{n^2+1}{n} \right\rangle$ diverges.

Notice that $\frac{n^2+1}{n} \approx \frac{n^2}{n} = n$ which is unbounded. So it's safe to guess that our sequence is unbounded and we know that unbounded sequences diverge. Let's prove this.

Suppose $\left\langle \frac{n^2+1}{n} \right\rangle$ is bounded. Then there exists some B such that $\left| \frac{n^2+1}{n} \right| \leq B$ for all n. However, consider $n = B$. Then $\left| \frac{B^2+1}{B} \right| = \frac{B^2+1}{B} > \frac{B^2}{B} = B$. Contradiction! Therefore, the sequence is unbounded.

One of our theorems states that all convergent sequences are bounded. Therefore, any unbounded sequence cannot be convergent. Therefore, our sequence must diverge since it is not bounded.

(d) Let $a_n \to A$ and $b_n \to B$. Show that $a_n + b_n \to A + B$.

Let $\epsilon > 0$. Then since $a_n \to A$ there exists some N_a such that $|a_n - A| < \epsilon/2$ for all $n \geq N_a$. Likewise, there exists some N_b such that $|b_n - B| < \epsilon/2$ for all $n \geq N_b$. Let $N = \max\{N_a, N_b\}$. Suppose $n \geq N$. Then $|(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)| \leq |a_n-A|+|b_n-B| < \epsilon/2+\epsilon/2 = \epsilon$ since $n \geq N \geq N_a$ and $n \geq N \geq N_b$. Therefore, $(a_n + b_n) \to (A + B)$.

2. (20 points) Some set stuff.

(a) Let A, B, C, D be sets. Suppose that $A \cup B \subseteq C \cup D$, $A \cap B = \phi$, and $C \subseteq A$. Prove that $B \subseteq D$.

Our goal is to show that $B \subseteq D$, so we should begin with an element of B and then try to show it is an element of D.

Suppose $x \in B \Rightarrow x \in A \cup B \Rightarrow x \in C \cup D$ since $A \cup B \subseteq C \cup D$. Thus either $x \in C$ or $x \in D$. We need to rule out the former option. Suppose $x \in C$. Then $x \in A$ since $C \subseteq A$. But this would mean that $x \in B$ and $x \in A$ so that $x \in A \cap B$. Contradiction! This is impossible since $A \cap B = \phi$. Therefore, $x \notin C$, so $x \in D$. So all the elements of B are also elements of D. Therefore, $B \subseteq D$.

(b) Let $f : X \to Y$ and let $T \subseteq Y$. Prove that if f is onto, then $f(f^{-1}(T)) = T$. Point out which "half" of your proof does **not** need the "onto" hypothesis.

To show set equality we need to show containment in both directions. All elements in $f(f^{-1}(T))$ are in T and vice-versa.

- Suppose $y \in f(f^{-1})(T)$. This implies that there exists some $x \in f^{-1}(T)$ such that $f(x) = y$. But $x \in f^{-1}(T)$ means that $f(x) \in T$. Therefore, $y = f(x) \in T$ and so $f(f^{-1}(T)) \subseteq T$.
- Suppose $y \in T$. We need y to be in the image of f. At this point we must use our "onto" assumption. Since f is onto, there exists some $x \in X$ such that $f(x) = y$. But notice that $f(x) = y \in T$. Therefore, $x \in f^{-1}(T)$. So we have shown that $y = f(x) \in f(f^{-1}(T))$. Thus $T \subseteq f(f^{-1}(T))$.

Finally, containment both ways implies equality.

Note: The onto assumption is necessary. Consider the map $f : \{a\} \to \{1, 2\}$ such that $f(a) = 1$. Then $f^{-1}(\{2\}) = \phi$ (nothing maps to 2), so $f(f^{-1}(\{2\}) = f(\phi) = \phi \neq \{2\}$.

3. (28 points) For each of the following functions, decide if f is 1-1, onto, both, or neither. **Prove your answers!**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 5x + 2$

Injective Suppose $f(x) = f(y)$. Then $5x + 2 = 5y + 2$ so that $5x = 5y$ and thus $x = y$. Hence, f is one-to-one.

Surjective Suppose $y \in \mathbb{R}$. [Scratch work: $5x + 2 = y \Rightarrow 5x = y - 2 \Rightarrow x = (y - 2)/5$] Consider $x = (y - 2)/5 \in \mathbb{R}$. Then $f(x) = f((y - 2)/5) = 5[(y - 2)/5] + 2 = (y - 2) + 2 = y$. Therefore, f is onto.

f is both one-to-one and onto. Thus f is a bijection.

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 5x + 2$

Injective Same as in part (a).

Not Surjective Using our previous scratch work we can see that the equation $5x + 2 = y$ can't always be solved for $x \in \mathbb{Z}$. In particular, it doesn't work when $y = 0$.

Suppose $f(x) = 0$. Then $5x + 2 = 0$ so $5x = -2$ and thus $x = -2/5 \notin \mathbb{Z}$. Therefore, $f(x) \neq 0$ for all $x \in \mathbb{Z}$. Thus 0 is not in the range of $\mathbb{Z}$ and so f is not onto.

f is one-to-one, but not onto.

(c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} 2x & x \text{ is even} \\ x + 1 & x \text{ is odd} \end{cases}$

Let's make a table of values of f to get a feel for how the function works.

$x =$	-2	-1	0	1	2
$f(x) =$	-4	0	0	2	4

Not Injective Our table shows that f is not one-to-one. In particular, $f(-1) = 0 = f(0)$ but $-1 \neq 0$.

Not Surjective The table also leads us to believe the function isn't onto. In fact, no negative numbers get hit.

Suppose $f(x) = 1$. Then if x is even we have that $f(x) = 2x = 1$ so $x = 1/2$. Contradiction! (1/2 isn't an integer.) Thus x must be odd, so $f(x) = x+1 = 1$ which implies that $x = 0$. Contradiction! (0 is even.) Therefore, there is no x such that $f(x) = 1$ and so 1 is not in the range of f. Therefore, f is not onto.

f is neither one-to-one nor onto.

(d) Let $g : X \to Y$ and $f : Y \to Z$. Assume that $f \circ g : X \to Z$ is a bijection. If f **must be** 1-1, prove it. If f might not be 1-1, give a counter-example. If f **must be** onto, prove it. If f might not be onto, give a counter-example. [A counter-example should specify **both** f and g and be accompanied by a proof that it is in fact a counter-example.]

Not Injective Consider $f : \{1, 2\} \to \{1\}$ where $f(1) = f(2) = 1$ and $g : \{1\} \to \{1, 2\}$ where $g(1) = 1$. Then $f \circ g : \{1\} \to \{1\}$ is defined by $(f \circ g)(1) = f(g(1)) = f(1) = 1$. Notice that $f \circ g$ is the identity function on the set $\{1\}$. $f \circ g$ is clearly bijective.

Notice that f is onto, but not one-to-one since $f(1) = 1 = f(2)$. f may fail to be injective.

Surjective Suppose $z \in Z$. Now $f \circ g$ is bijective. Thus $f \circ g$ is onto. So there exists some $x \in X$ such that $(f \circ g)(x) = f(g(x)) = z$. Taking $y = g(x) \in Y$ we have $f(y) = z$. Thus z is in the image of f and so it is onto.

f must be onto, but can fail to be one-to-one.

4. (28 points) Equivalent Nonsense.

Recall that for integers a and b, $a \cong b \pmod 4$ if and only if there exists some $k \in \mathbb{Z}$ such that $a = b + 4k$.

(a) Prove that $a \cong b \pmod 4$ is an equivalence relation on $\mathbb{Z}$.

Reflexive Suppose $a \in \mathbb{Z}$. Then $a = a + 4(0)$. Thus $a \cong a \pmod 4$.

Symmetric Suppose $a, b \in \mathbb{Z}$ and $a \cong b \pmod 4$. Then there exists some $k \in \mathbb{Z}$ such that $a = b + 4k$. So $a - 4k = b$ and thus $b = a + 4(-k)$ (of course $-k \in \mathbb{Z}$ since $k \in \mathbb{Z}$). Therefore, $b \cong a \pmod 4$.

Transitive Suppose $a, b, c \in \mathbb{Z}$, $a \cong b \pmod 4$, and $b \cong c \pmod 4$. Then there exits $k, \ell \in \mathbb{Z}$ such that $a = b + 4k$ and $b = c + 4\ell$. Thus $a = b + 4k = (c + 4\ell) + 4k = c + 4(k + \ell)$ (of course $k + \ell \in \mathbb{Z}$). Therefore, $a \cong c$ (mod).

(b) List the equivalence classes of this equivalence relation.

- $[0] = \{\dots, -8, -4, 0, 4, 8, \dots\} = \{4k \mid k \in \mathbb{Z}\}$
- $[1] = \{\dots, -7, -3, 1, 5, 9, \dots\} = \{4k + 1 \mid k \in \mathbb{Z}\}$
- $[2] = \{\dots, -6, -2, 2, 6, 10, \dots\} = \{4k + 2 \mid k \in \mathbb{Z}\}$
- $[3] = \{\dots, -5, -1, 3, 7, 11, \dots\} = \{4k + 3 \mid k \in \mathbb{Z}\}$

(c) Prove that the function $f : \mathbb{Z}_4 \to \mathbb{Z}_{10}$ defined by $f([n]) = [5n]$ is well-defined. [$\mathbb{Z}_m$ are the equivalence classes of integers mod m.]

Well defined means that equivalent inputs give equivalent outputs. Suppose $[m] = [n] \in \mathbb{Z}_4$ (equal inputs). Then since $[m] = [n]$, $m \cong n \pmod 4$, so there exists some $k \in \mathbb{Z}$ such that $m = n + 4k \Longrightarrow 5m = 5n+20k \Longrightarrow 5m = 5n+10(2k)$. Therefore, $5m \cong 5n \pmod{10}$ and so $f([m]) = [5m] = [5n] = f([n])$ (equal outputs). Therefore, f is well-defined.

Note: If out map had been $f([n]) = [6n]$, this argument would fail. In fact, $[0] = [4]$ in $\mathbb{Z}_4$, but $f([0]) = [0]$ is not equal to $f([4]) = [24] = [4]$ in $\mathbb{Z}_{10}$.

(d) Let $n \in \mathbb{Z}$ be a sum of squares. [That is there exist integers a and b such that $n = a^2 + b^2$.] Prove that $n \ncong 3 \pmod 4$.

Recall that addition and multiplication modulo 4 are well-defined. So we can swap out any integer with another integer as long as they are equal mod 4.

Suppose $n = a^2 + b^2$ for some $a, b \in \mathbb{Z}$. Then $a, b \cong 0,\ 1,\ 2,$ or $3 \pmod 4$. Note that $0^2, 1^2, 2^2, 3^2 = 0, 1, 4, 9 \cong 0, 1, 0, 1 \pmod 4$. Thus $a^2 + b^2 \cong 0^2 + 0^2, 0^2 + 1^2, 1^2 + 0^2,$ or $1^2 + 1^2 \cong 0, 1,$ or $2 \pmod 4$. Therefore, it is impossible for n to be equivalent to 3 modulo 4 if n is the sum of squares.