Math 2510 | # Test #2 | **April 15$^{\text{th}}$, 2015**

Name: Answer Key **Be sure to show your work!**

1. (30 points) Converging Questions

(a) Prove that $\left\langle \frac{2n^3+n}{n^3+3} \right\rangle$ converges.

Let $\varepsilon > 0$. Set $N = \max\left\{ \left\lceil \frac{1}{\varepsilon} \right\rceil, 6 \right\}$. Notice that for $n \geq N$ we have $n \geq 6$ so $|n-6| = n-6 < n$. For all $n \geq N$,

$$\left| \frac{2n^3+n}{n^3+3} - 2 \right| \leq \left| \frac{2n^3+n}{n^3+3} - 2\frac{n^3+3}{n^3+3} \right| \leq \left| \frac{n-6}{n^3+3} \right| = \frac{n-6}{n^3+3} < \frac{n}{n^3+3} < \frac{n}{n^3} = \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{1}{1/\varepsilon} = \varepsilon$$

Hence, by the definition of convergence, we have that $\left\langle \frac{2n^3+n}{n^3+3} \right\rangle$ converges to 2. $\square$

(b) Prove that $\left\langle \frac{(-1)^n n}{n^2+1} \right\rangle$ converges.

Let $\varepsilon > 0$. Set $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Notice that for $n \geq N$ we have

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| = \left| \frac{(-1)^n n}{n^2+1} \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{N} \leq \frac{1}{1/\varepsilon} = \varepsilon$$

Hence, by the definition of convergence, we have that $\left\langle \frac{(-1)^n n}{n^2+1} \right\rangle$ converges to 0. $\square$

(c) Prove that $\left\langle \frac{n^4}{n-1} \right\rangle_{n=2}^{\infty}$ diverges.

Suppose by way of contradiction that $\left\langle \frac{n^4}{n-1} \right\rangle_{n=2}^{\infty}$ is bounded by say M. Then for every $n \geq 2$, we have

$$\frac{n^4}{n-1} \leq M.$$

Notice that $n^4/(n-1) > 0$ so that $M > 0$. Consider $n = \lceil M \rceil + 1$ (since $\lceil M \rceil \geq 1$ because $M > 0$, $n \geq 2$). We have

$$\frac{n^4}{n-1} = \frac{(\lceil M \rceil + 1)^4}{(\lceil M \rceil + 1) - 1} = \frac{(\lceil M \rceil + 1)^4}{\lceil M \rceil} > \frac{M^4}{M} = M^3 > M,$$

a contradiction. So we must have that $\left\langle \frac{n^4}{n-1} \right\rangle_{n=2}^{\infty}$ is unbounded. As such, it can not be convergent. $\square$

(d) Let $a_n \to L$ (for some $L \in \mathbb{R}$) and $b_n \to 0$. Show that $a_n b_n \to 0$.

As $a_n \to L$, we must have that a_n is bounded by some number say M, so that for all n, $|a_n| \leq M$. Now, as $b_n \to 0$, we must have that for any $\varepsilon > 0$, there exists some N such that for all $n \geq N$, we have $|b_n - 0| = |b_n| < \varepsilon$. In particular, for $(\varepsilon/M) > 0$, there must exist some N, call it N_0, such that for all $n \geq N_0$, $|b_n| < (\varepsilon/M)$. Note then that for all $n \geq N_0$ we have

$$|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq M|b_n| < M\frac{\varepsilon}{M} = \varepsilon.$$

Hence, by the definition of convergence, we have that $a_n b_n \to 0$. $\square$

2. (20 points) Some set stuff.

(a) Let A, B, C be sets. Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

To show two sets are equal, we must show containment both ways.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. So, $x \in B$ or $x \in C$. Hence, we must have that $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. That is, $x \in A \cap B$ or $x \in A \cap C$; regardless, $x \in (A \cap B) \cup (A \cap C)$. We have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Let $x \in (A \cap B) \cup (A \cap C)$. So $x \in A \cap B$ or $x \in A \cap C$. Thus x must be an element of A and must either be an element of B or C. So, $x \in A$ and $x \in B \cup C$ and thence, $x \in A \cap (B \cup C)$. We have $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$.

Quick proof: $x \in A \cap (B \cup C) \iff x \in A$ and $x \in B \cup C \iff x \in A$ and $(x \in B$ or $x \in C) \iff x \in A$ and $x \in B$ or $x \in A$ and $x \in C \iff x \in A \cap B$ or $x \in A \cap C \iff x \in (A \cap B) \cup (A \cap C)$.

We conclude $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

(b) Let $f : X \to Y$ be a function and $T_1, T_2 \subseteq Y$. Show that $f^{-1}(T_1 - T_2) = f^{-1}(T_1) - f^{-1}(T_2)$.

Keep in mind that $x \in f^{-1}(S)$ if and only if $f(x) \in S$ since $f^{-1}(S)$ is the set of all elements which map to S.

$x \in f^{-1}(T_1 - T_2) \iff f(x) \in T_1 - T_2 \iff f(x) \in T_1$ and $f(x) \notin T_2 \iff x \in f^{-1}(T_1)$ and $x \notin f^{-1}(T_2) \iff x \in f^{-1}(T_1) - f^{-1}(T_2)$.

We conclude $f^{-1}(T_1 - T_2) = f^{-1}(T_1) - f^{-1}(T_2)$.

(c) Let $f : X \to Y$ and let $R_1, R_2 \subseteq X$. Prove that if f is one-to-one, then $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$. Point out which "half" of your proof does **not** need the "one-to-one" hypothesis.

Let $y \in f(R_1 \cap R_2)$. Then there exists an $x \in R_1 \cap R_2$ such that $f(x) = y$. So $x \in R_1$ and $x \in R_2$. Hence $f(x) = y \in f(R_1)$ and $f(x) = y \in f(R_2)$. We have $f(x) \in f(R_1) \cap f(R_2)$ and that $f(R_1 \cap R_2) \subseteq f(R_1) \cap f(R_2)$.

Let $y \in f(R_1) \cap f(R_2)$. Then there exist $x_1 \in R_1$ and $x_2 \in R_2$ such that $f(x_1) = f(x_2) = y$. **Because f is one-to-one** we may conclude that $x_1 = x_2$. Further, $x_1 \in R_1$ and $x_1 = x_2 \in R_2$ so that $x \in R_1 \cap R_2$. Hence $f(x_1) = y \in f(R_1 \cap R_2)$. We have that $f(R_1 \cap R_2) \supseteq f(R_1) \cap f(R_2)$.

We conclude $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$. □

3. (25 points) For each of the following functions, decide if f is 1-1, onto, both, or neither. **Prove your answers!**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$

BOTH.

One-to-one: Suppose $f(x_1) = f(x_2)$ then $x_1^3 + 1 = x_2^3 + 1$. From this we see $x_1^3 = x_2^3$, and because the cube root of a number is unique we have that $x_1 = x_2$.

Onto: We show that any element in the codomain can be mapped to. Let $y \in \mathbb{R}$. Then $\sqrt[3]{y-1} \in \mathbb{R}$ as well and we have $f(\sqrt[3]{y-1}) = (\sqrt[3]{y-1})^3 + 1 = y - 1 + 1 = y$. □

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$

NEITHER.

Not one-to-one: Consider $f(2)$ and $f(-2)$. We have $f(2) = f(-2) = 4$ but $2 \neq -2$.

Not onto: Suppose that $f(x) = -1$. Then $x^2 = -1$ and so $x = \sqrt{-1} \notin \mathbb{Z}$. Therefore, -1 isn't in the range of f. □

(c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} \dfrac{x}{2} & x \text{ is even} \\ \dfrac{x-1}{2} & x \text{ is odd} \end{cases}$

ONLY ONTO.

Not one-to-one: Note that $f(7) = (7-1)/2 = 3$ and $f(6) = 6/2 = 3$ but $6 \neq 7$. □

Onto: Consider an arbitrary integer y. Then $2y \in \mathbb{Z}$ is even and so $f(2y) = 2y/2 = y$ (using the formula for *even* inputs). □

(d) Show that composing two one-to-one functions yields a one-to-one function and that composing two onto functions yields and onto function. Specifically, let $f : X \to Y$ and $g : Y \to Z$. Show that $g \circ f$ is one-to-one if we assume both f and g are one-to-one. Then show $g \circ f$ is onto if we assume both f and g are onto.

One-to-one: Suppose $g(f(x_1)) = g(f(x_2))$. Becuase g is one-to-one, we must have that $f(x_1) = f(x_2)$. Because f is one-to-one, $x_1 = x_2$. We conclude $g \circ f$ is one-to-one. □

Onto: Consider an arbitrary element in $z \in Z$. Since g is onto there must exist some $y \in Y$ such that $g(y) = z$. As f is onto there exists an $x \in X$ such that $f(x) = y$. Therefore, $g(f(x)) = g(y) = z$. We conclude $g \circ f$ onto. □

4. (25 points) Equivalent Nonsense.

(a) Give the definition of an equivalence relation (with details). Then consider $x \sim y$ only if $x \geq y$. This is a relation on $\mathbb{R}$. Why isn't it an equivalence relation?

An equivalence relation on a set X is a relation $\sim$ which is reflexive, symmetric and transitive. Recall that a relation is

- *reflexive* if $a \sim a$ for all $a \in X$;
- *symmetric* if $a \sim b$ then $b \sim a$ for all $a, b \in X$;
- *transitive* if $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in X$;

The relation in question here is not an equivalence relation becauses it is **not symmetric**. For example: $2 \geq 1$ but $1 \not\geq 2$. (On the other hand, it is reflexive: $x \geq x$ and transitive: $x \geq y$ and $y \geq z$ implies $x \geq z$.)

(b) List the equivalence classes of the relation "mod 5" (on $\mathbb{Z}$).

$$\begin{aligned}
[0] &= \{k \mid \exists j \in \mathbb{Z} \text{ such that } k = 5j\} &&= \{\dots, -10, -5, 0, 5, 10, \dots\} \\
[1] &= \{k \mid \exists j \in \mathbb{Z} \text{ such that } k = 5j+1\} &&= \{\dots, -9, -4, 1, 6, 11, \dots\} \\
[2] &= \{k \mid \exists j \in \mathbb{Z} \text{ such that } k = 5j+2\} &&= \{\dots, -8, -3, 2, 7, 12, \dots\} \\
[3] &= \{k \mid \exists j \in \mathbb{Z} \text{ such that } k = 5j+3\} &&= \{\dots, -7, -2, 3, 8, 13, \dots\} \\
[4] &= \{k \mid \exists j \in \mathbb{Z} \text{ such that } k = 5j+4\} &&= \{\dots, -6, -1, 4, 9, 14, \dots\}
\end{aligned}$$

(c) Prove that the function $f : \mathbb{Z}_6 \to \mathbb{Z}_{22}$ defined by $f([n]) = [11n]$ is well-defined. Then show that $g : \mathbb{Z}_6 \to \mathbb{Z}_{10}$ "defined" by $g([n]) = [11n]$ is not well-defined. [$\mathbb{Z}_m$ are the equivalence classes of integers mod m.]

Consider $[x], [y] \in \mathbb{Z}_6$. Then $x = y + 6k$ for some $k \in \mathbb{Z}$. This implies that $11x = 11y + 66k$ so $11x = 11y + 22(3k)$. Therefore, $f([x]) = [11x] = [11y] = f([y])$ in $\mathbb{Z}_{22}$. Hence, f is well defined.

Note $[1] = [7]$ in $\mathbb{Z}_6$ and that $g([1]) = [11] = [1]$ in $\mathbb{Z}_{10}$ and $g([7]) = [77] = [7]$ in $\mathbb{Z}_{10}$. However, $g([1]) = [1] \neq [7] = g([7])$ in $\mathbb{Z}_{10}$. So g is not well defined (equivalent inputs do *not* yield equivalent outputs).

(d) Let $x \in \mathbb{Z}_{>0}$ (a positive integer). We can write x in decimal form: $x = d_\ell \cdots d_1 d_0$ (d_j is the j^{th} digit). Actually, $x = d_\ell \cdot 10^\ell + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$ (where $d_j \in \{0, \dots, 9\}$).

Explain why x is divisible by 3 if and only if $d_\ell + \cdots + d_0$ (the sum of its digits) is divisible by 3. [*Hint:* mod 3]

Recall that x is divisible by 3 if and only if $x \equiv 0 \pmod 3$. So consider $x = d_\ell \cdot 10^\ell + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$ (where $d_j \in \{0, \dots, 9\}$). Then

$$\begin{aligned}
x &\equiv d_\ell \cdot 10^\ell + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0 \pmod 3 \\
&\equiv d_\ell \cdot 1^\ell + \cdots + d_2 \cdot 1^2 + d_1 \cdot 1 + d_0 \pmod 3 \\
&\equiv d_\ell + \cdots + d_2 + d_1 + d_0 \pmod 3
\end{aligned}$$

Since $10 \equiv 1 \pmod 3$, we could replace each 10^j with $1^j = 1$ in the above calculation. Therefore, $x \equiv 0 \pmod 3$ if and only if $d_\ell + \cdots + d_2 + d_1 + d_0 \equiv 0 \pmod 3$. This means that x is divisible by 3 if and only if $d_\ell + \cdots + d_1 + d_0$ is divisible by 3. □