Name:

Be sure to show your work!

- 1. (30 points) Converging Questions
- (a) Prove that $\left\langle \frac{2n^3 + n}{n^3 + 3} \right\rangle$ converges.

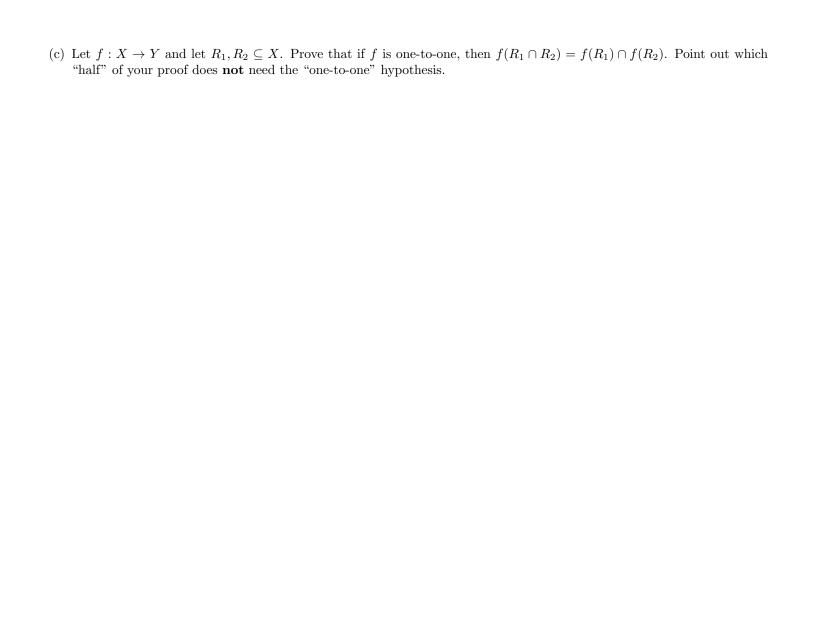
(b) Prove that $\left\langle \frac{(-1)^n n}{n^2 + 1} \right\rangle$ converges.

(c) Prove that $\left\langle \frac{n^4}{n-1} \right\rangle_{n=2}^{\infty}$ diverges.

(d) Let $a_n \to L$ (for some $L \in \mathbb{R}$) and $b_n \to 0$. Show that $a_n b_n \to 0$.

- 2. (20 points) Some set stuff.
- (a) Let A, B, C be sets. Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) Let $f: X \to Y$ be a function and $T_1, T_2 \subseteq Y$. Show that $f^{-1}(T_1 - T_2) = f^{-1}(T_1) - f^{-1}(T_2)$.



| 3. | (25) | points) | For each of the following for | functions, decide if | f is 1-1, onto, | both, or neither. | Prove your answers |
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(a) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$

(b) Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$

(c) Let
$$f: \mathbb{Z} \to \mathbb{Z}$$
 be defined by $f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ \frac{x-1}{2} & x \text{ is odd} \end{cases}$

(d) Show that composing two one-to-one functions yields a one-to-one function and that composing two onto functions yields and onto function. Specifically, let $f: X \to Y$ and $g: Y \to Z$. Show that $g \circ f$ is one-to-one if we assume both f and g are one-to-one. Then show $g \circ f$ is onto if we assume both f and g are onto.

| 4. (25 points) Equivalent Nonsense. |
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| (a) Give the definition of an equivalence relation (with details). Then consider $x \sim y$ only if $x \geq y$. This is a relation on \mathbb{R} . Why isn't it an equivalence relation? |
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| (b) List the equivalence classes of the relation "mod 5" (on \mathbb{Z}). |
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| (c) | Prove that the function $f: \mathbb{Z}_6 \to \mathbb{Z}_{22}$ defined by $f([n]) = [11n]$ is well-defined. Then show that $g: \mathbb{Z}_6 \to \mathbb{Z}_{10}$ "defined" by $g([n]) = [11n]$ is not well-defined. $[\mathbb{Z}_m]$ are the equivalence classes of integers mod m .] | |
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| (d) | Let $x \in \mathbb{Z}_{>0}$ (a positive integer). We can write x in decimal form: $x = d_{\ell} \cdots d_1 d_0$ (d_j is the j^{th} digit). Actually, $x = d_{\ell} \cdot 10^{\ell} + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$ (where $d_j \in \{0, \dots, 9\}$). Explain why x is divisible by 3 if and only if $d_{\ell} + \cdots + d_0$ (the sum of its digits) is divisible by 3. [Hint: mod 3] | |
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