Math 2510 **Test #2** **April 15^{th}, 2015**

Name: ________________________________ **Be sure to show your work!**

1. (30 points) Converging Questions

(a) Prove that $\left\langle \frac{2n^3+n}{n^3+3} \right\rangle$ converges.

(b) Prove that $\left\langle \frac{(-1)^n n}{n^2+1} \right\rangle$ converges.

(c) Prove that $\left\langle \frac{n^4}{n-1} \right\rangle_{n=2}^{\infty}$ diverges.

(d) Let $a_n \to L$ (for some $L \in \mathbb{R}$) and $b_n \to 0$. Show that $a_n b_n \to 0$.

2. (20 points) Some set stuff.

(a) Let A, B, C be sets. Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(b) Let $f : X \to Y$ be a function and $T_1, T_2 \subseteq Y$. Show that $f^{-1}(T_1 - T_2) = f^{-1}(T_1) - f^{-1}(T_2)$.

(c) Let $f : X \to Y$ and let $R_1, R_2 \subseteq X$. Prove that if f is one-to-one, then $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$. Point out which "half" of your proof does **not** need the "one-to-one" hypothesis.

3. (25 points) For each of the following functions, decide if f is 1-1, onto, both, or neither. **Prove your answers!**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$

(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$

(c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} \dfrac{x}{2} & x \text{ is even} \\ \dfrac{x-1}{2} & x \text{ is odd} \end{cases}$

(d) Show that composing two one-to-one functions yields a one-to-one function and that composing two onto functions yields and onto function. Specifically, let $f : X \to Y$ and $g : Y \to Z$. Show that $g \circ f$ is one-to-one if we assume both f and g are one-to-one. Then show $g \circ f$ is onto if we assume both f and g are onto.

4. (25 points) Equivalent Nonsense.

(a) Give the definition of an equivalence relation (with details). Then consider $x \sim y$ only if $x \geq y$. This is a relation on $\mathbb{R}$. Why isn't it an equivalence relation?

(b) List the equivalence classes of the relation "mod 5" (on $\mathbb{Z}$).

(c) Prove that the function $f : \mathbb{Z}_6 \to \mathbb{Z}_{22}$ defined by $f([n]) = [11n]$ is well-defined. Then show that $g : \mathbb{Z}_6 \to \mathbb{Z}_{10}$ "defined" by $g([n]) = [11n]$ is not well-defined. [$\mathbb{Z}_m$ are the equivalence classes of integers mod m.]

(d) Let $x \in \mathbb{Z}_{>0}$ (a positive integer). We can write x in decimal form: $x = d_\ell \cdots d_1 d_0$ (d_j is the j^{th} digit). Actually, $x = d_\ell \cdot 10^\ell + \cdots + d_2 \cdot 10^2 + d_1 \cdot 10 + d_0$ (where $d_j \in \{0, \ldots, 9\}$).

Explain why x is divisible by 3 if and only if $d_\ell + \cdots + d_0$ (the sum of its digits) is divisible by 3. [*Hint:* mod 3]