Math 4710/5710 **Set Test — In Class** **September 15^{th}, 2014**

Name: ____________________________________ **Be sure to show your work!**

1. (20 points) Memorization

(a) Let T be a (non-empty) set with relation $<$. What properties must $(T, <)$ possess so that it is *totally ordered* (i.e. simply ordered)? Don't just state these properties. Define the properties themselves – give details!

(b) State *Zorn's Lemma.*

2. (20 points) A few proofs

(a) Let $f : X \to Y$ and suppose that $A, B \subseteq Y$. Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) Let $f : X \to Y$ and suppose that $A, B \subseteq X$. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) Let $A_i \subseteq X$ for all $i \in I$ and $B \subseteq X$. Show that $\left(\bigcup_{i \in I} A_i\right) \cap B = \bigcup_{i \in I} (A_i \cap B)$.
Note: I is an arbitrary non-empty index set. Do not assume it is finite or countable.

3. (20 points) A proofs about functions.

(a) Suppose $f : A \to B$ and $g : B \to C$ are injections. Show that $g \circ f : A \to C$ is an injection as well.

(b) Suppose $f : A \to B$ and $g : B \to C$ are surjections. Show that $g \circ f : A \to C$ is a surjection as well.

(c) Prove that $|\mathcal{P}(X)| = |2^X|$ where $\mathcal{P}(X)$ is the power set of X and 2^X is the set of all functions from X to $\{0, 1\}$. Do not assume anything about the set X (it could be finite, infinite, or even empty).

4. (20 points) Put the following sets in order from smallest to largest. Use $<$ and $=$ to indicate relationships. For example: $|A| = |B| < |C| < |D| = |E| = |F| = |G|$. [By the way, this is not the correct answer.] Justify your answer!

- $A = \{n \in \mathbb{Z} \mid \text{n is even and } n < -999\}$
- $B = \{x \in \mathbb{C} \mid x \text{ is a solution of } x^{15} - \sqrt{6}x^9 + 4x^8 - x^5 + 17x + \pi = 0\}$
- $C = \mathcal{P}(B)$
- $D = \mathbb{R}^{\mathbb{Z}}$ (functions from $\mathbb{Z}$ to $\mathbb{R}$)
- $E = \mathbb{R}^{\mathbb{R}}$ (functions from $\mathbb{R}$ to $\mathbb{R}$)
- $F = \{X \subseteq \mathbb{Z} \mid X \text{ is not finite }\}$
- $G = \mathbb{Q} \times \mathbb{Q}$

5. (20 points) Everyone's favorite...True, Possible, or False!?! If a statement is always true, prove that it is always true. If a statement is always false, show that it cannot be true. If a statement is true in some cases and false in others, give an example of it holding and an example of it failing to hold.

(a) Let $X \subseteq \{n \in \mathbb{N} \mid n \text{ is prime}\}$.

Is it **True** / **Possible** / **False** that $\varnothing \in \mathcal{P}(X)$ and $\varnothing \subseteq \mathcal{P}(X)$?

(b) Let $X \subseteq \mathbb{R}$ and $f : X \to X$ be defined by $f(x) = 2x$.

Is it **True** / **Possible** / **False** that f is onto?

(c) Let X be a set.

Is it **True** / **Possible** / **False** that $\mathcal{P}(X)$ is countably infinite?

(d) Let $\prod_{i \in I} B_i \subseteq \prod_{i \in I} A_i$ (Recall that Π = cartesian product)

Is it **True** / **Possible** / **False** that $B_i \subseteq A_i$ for all $i \in I$?

- This portion of the exam must be turned in **no later** than 12:00pm on Wednesday, September 17^{th}, 2014.
- You may use notes, textbooks, and existent online resources to complete these problem.
- You may **not** ask anyone (except me) for help.

6. (30 points) Rework the test.

7. (10 points) [5710 – Grad Problem] Let $\mathcal{A}$ be a well-ordered set. Let $\mathcal{A}_0 \subseteq \mathcal{A}$. We say $\mathcal{A}_0$ is *inductive* if for every $\alpha \in \mathcal{A}$ we have that $S_\alpha = \{\beta \in \mathcal{A} \mid \beta < \alpha\} \subseteq \mathcal{A}_0$ implies that $\alpha \in \mathcal{A}_0$. Prove the *principle of transfinite induction*: If $\mathcal{A}$ is well-ordered and $\mathcal{A}_0$ is an inductive subset of $\mathcal{A}$, then $\mathcal{A}_0 = \mathcal{A}$. Also, discuss how this is related to our familiar "mathematical induction".

8. (10 points) [5710 – Grad Problem] Explain why $X = \{1,2\} \times \mathbb{Z}_{>0}$ and $Y = \mathbb{Z}_{>0} \times \{1,2\}$ are well-ordered when we give $\{1,2\}$ and $\mathbb{Z}_{>0}$ their usual orders and X, Y dictionary orders. Do these sets have the same order type? That is – is there a bijection between X and Y which preserves orders (i.e. a bijection $\varphi : X \to Y$ such that $\mathbf{v} < \mathbf{w}$ iff $f(\mathbf{v}) < f(\mathbf{w})$)?