

# Types of Infinity

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## Abstract

The concept of infinity pervades much of modern mathematics. Even elementary school students use the word infinity. However, what do we mean by infinity? While, for many infinity merely means either very large or not finite, in various contexts, the term infinity carries more information. Here we consider infinity in the contexts of cardinality (sizes of sets), ordinals (the order of numbers), and real numbers (a number system). Our aim is to introduce these various notions of infinity to interested mathematics students and whet their appetite for more. To accomplish this, we have included three online applets for readers to consider when teaching or learning about mappings from between natural and rational numbers.

## 1 Introduction<sup>1</sup>

Teachers and students of mathematics can appreciate Buzz Lightyear’s mathematical prowess and insight as he correctly states, “To infinity and beyond.” Indeed, as we will see, Buzz gets it!

Even in elementary school, students use the word *infinity* to denote the notion of very large. Later in mathematics, students discuss infinite sets as well as infinities related to number systems. But, do we all mean the same thing by *infinity*? As we will see, infinity has a number of distinct yet interconnected meanings. For instance, infinity can be explored in the contexts of cardinals (sizes of sets), ordinals (orderings of sets), and real numbers (a number system). In this paper, we present these ideas in a manner commensurate with the interests of inquisitive high school and undergraduate students and teachers. We hope that these ideas regarding infinity stimulate the reader to new interests in these intriguing concepts.

This paper is a soft investigation of concepts regarding infinity. For more detailed exposition, consider the following: [Halmos, 2011] and [Roitman, 1990]. In our companion paper [Cook et al., 2020] we consider the concept of  $\infty$  and employ one- and two-point compactifications to investigate how the graphs of rational functions can pass through or bounce off  $\infty$  in a manner similar to the graph of a polynomial passing through or bouncing off the  $x$ -axis.

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<sup>1</sup>In order to simplify several discussions, we freely assume the Axiom of Choice throughout this paper. This axiom is detailed later in this paper.

## 2 Cardinal Numbers

Most simply stated, the cardinality of a set is the number of elements in that set. Let us begin with a definition which will be followed up with a couple of examples.

**Definition 2.1.** *Let  $A$  and  $B$  be sets. We say that  $A$  and  $B$  have the same cardinality, denoted  $\text{card}(A) = \text{card}(B)$ , if and only if there is a one-to-one and onto function (i.e., invertible function) whose domain is  $A$  and range is  $B$ .*

A function  $f : A \rightarrow B$  is one-to-one if each element of  $A$  maps to a unique element of  $B$ , though there may be some extra elements of  $B$  which are not partnered with elements of  $A$ . A function  $f : A \rightarrow B$  is onto if each element of  $B$  is mapped to by an element of  $A$ , though there may be some elements of  $B$  which are partnered with several elements of  $A$ . A function is invertible if it is both one-to-one and onto. Such a function makes sure every element of  $A$  is uniquely partnered with some element of  $B$  and vice-versa.

This means that sets of objects have the same size (i.e., cardinality) if we can match the elements between the sets in a one-to-one fashion. For example,  $P = \{a, b, c\}$  and  $Q = \{\odot, \ominus, \otimes\}$  both have the same cardinality:  $\text{card}(P) = \text{card}(Q)$  because we can exhibit an invertible function between these sets:  $a \leftrightarrow \odot$ ,  $b \leftrightarrow \ominus$ ,  $c \leftrightarrow \otimes$ . More simply stated: every element in set  $P$  maps to a distinct element in set  $Q$  and every element in  $Q$  maps to a distinct element in  $P$  (resulting in no unused elements in either direction). In this case, we might write  $\text{card}(P) = \text{card}(Q) = 3$ . Before moving on, let us be careful to point out that so far  $\text{card}(A)$  does not mean anything by itself. We define what it means for cardinalities to be the same and below we will define how one can compare cardinalities. But  $\text{card}(A)$  itself is not any kind of concrete object, yet.

When cardinalities are not the same, we can still compare them with inequalities. We write that  $\text{card}(A) \leq \text{card}(B)$  if there is a function  $f : A \rightarrow B$  which is one-to-one but possibly not onto. If there is a one-to-one function  $f : A \rightarrow B$  but there is no invertible function between  $A$  and  $B$ , we write  $\text{card}(A) < \text{card}(B)$ . We might want to restate this as: If set  $B$  has at least one more element than set  $A$ , then  $\text{card}(A) < \text{card}(B)$ . However, while this is true for finite sets, it fails for infinite sets. When a set is infinite, adding in one more element does not change its cardinality!

It is easy to show<sup>2</sup> that if one has an onto function  $f : A \rightarrow B$ , then there must be a one-to-one function  $g : B \rightarrow A$ . This means that one-to-one and onto are dual to each other. An intuitive, yet a bit tricky to prove, theorem says that if  $\text{card}(A) \leq \text{card}(B)$  and  $\text{card}(B) \leq \text{card}(A)$ , then  $\text{card}(A) = \text{card}(B)$  (i.e., if neither set is bigger than the other, they must have the same size). This result is known as the Cantor-Schröder-Bernstein theorem. It was published by Cantor without proof and then later proven by Schröder and then reproven independently by Bernstein.

To continue investigating the notion of infinite cardinalities, we must first define the relation of proper subset. Set  $A$  is a proper subset of set  $B$ , denoted  $A \subsetneq B$  if every every element of  $A$  is an element of  $B$  and there is at least one element in  $B$  which is not in  $A$ . For finite sets, this would automatically imply that  $\text{card}(A) < \text{card}(B)$ . However, let us investigate this a little more. Consider the sets  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{E} = \{0, 2, 4, \dots\}$ . While it may be tempting to say, “ $\mathbb{N}$  has twice as many elements as  $\mathbb{E}$ , since  $\mathbb{E}$  has none of the odd numbers in  $\mathbb{N}$ ”, this may not be correct. Consider the the function  $f : \mathbb{N} \rightarrow \mathbb{E}$  defined by  $f(n) = 2n$ . Since  $f$  is invertible,

<sup>2</sup>For each  $x \in \text{range}(f) = B$  choose some  $g_x \in A$  such that  $f(g_x) = x$  (can be done since  $f$  is onto). Then  $g : B \rightarrow A$  given by  $g(x) = g_x$  is a one-to-one function (if  $g_x = g(y) = g_y$  then  $x = f(g_x) = f(g_y) = y$ ). Note that in proving this result, we used the Axiom of Choice.

$\text{card}(\mathbb{N}) = \text{card}(\mathbb{E})$ . However,  $\mathbb{E} \subsetneq \mathbb{N}$ . While this may seem like a contradiction, it leads directly to the *definition* of infinite sets.

**Definition 2.2.** *Suppose  $A \subsetneq B$  and there is a function  $f : A \rightarrow B$  which is both one-to-one and onto. Then  $B$  is an infinite set. Indeed, both  $A$  and  $B$  are infinite sets. In other words, a set is infinite when it has the same cardinality as one of its proper subsets.*

If we define cardinalities as objects, then we can define finite and infinite sets in a more intuitive way. If  $\text{card}(A)$  is defined to be an object called an ordinal number (discussed later), then  $A$  is finite (i.e., not infinite) if and only if  $\text{card}(A)$  is a natural number.

Let us consider another example to help solidify this notion. We create a mapping from  $\mathbb{N} = \{0, 1, 2, \dots\}$  to  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ . Immediately, we notice that  $\mathbb{N} \subsetneq \mathbb{Z}$ . Again, it may seem that  $\mathbb{Z}$  has more than twice as many elements as  $\mathbb{N}$ . But, again, this is not quite correct. Consider the function  $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$  between  $\mathbb{N}$  and  $\mathbb{Z}$ . It is not hard to show that  $f$  is invertible, so  $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$ . Therefore, by the definition, both  $\mathbb{N}$  and  $\mathbb{Z}$  are infinite sets.

Before we investigate another example, we propose another definition which will be helpful.

**Definition 2.3.** *We say that a set  $X$  is countable if  $\text{card}(X) \leq \text{card}(\mathbb{N})$ . When  $\text{card}(X) = \text{card}(\mathbb{N})$  we say  $X$  is countably infinite. When  $\text{card}(X) < \text{card}(\mathbb{N})$  we have that  $X$  is a finite set. If  $\text{card}(X) > \text{card}(\mathbb{N})$ , then  $X$  is uncountable.*

Notice that countable sets are exactly those which can be (in theory) listed off. If  $\text{card}(X) = \text{card}(\mathbb{N})$ , then there is an invertible function  $f : \mathbb{N} \rightarrow X$ . So for each  $x \in X$  there is a unique  $n \in \mathbb{N}$  such that  $f(n) = x$  ( $n$  exists because  $f$  is onto, it is unique because  $f$  is one-to-one). Thus we can write  $x = f(n) = x_n$ . In other words,  $X = \{x_0, x_1, x_2, \dots\}$ . Conversely, if we can list off a set  $X = \{x_0, x_1, x_2, \dots\}$ , then we get a one-to-one function  $f : X \rightarrow \mathbb{N}$  defined by  $f(x_n) = n$ , so that  $\text{card}(X) \leq \text{card}(\mathbb{N})$  (with equality if  $X$  is infinite).

Now let's consider another mapping which may seem even more impossible. Recall that the rational numbers are defined as  $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ . Is it possible that there is a one-to-one and onto function mapping between  $\mathbb{N}$  and  $\mathbb{Q}$ ? Since  $\mathbb{N}$  is sparse on the number line (i.e., for any  $n \in \mathbb{N}$ , there are no natural numbers between  $n$  and  $n+1$ ) and  $\mathbb{Q}$  is dense (i.e., for any distinct  $a, b \in \mathbb{Q}$ , there exists an infinite number of rational numbers between  $a$  and  $b$ ), it certainly seems improbable that such a mapping may exist. However, let us look at the figure below.



It would now be quite intuitive to believe that there is only one cardinality for all infinite sets. In order to investigate this further, let us define power sets. Given any set  $X$ , its power set,  $\mathcal{P}(X)$  is the set of all of the subsets of  $X$ . For instance, let  $X = \{a, b, c\}$ . Then  $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$  where  $\emptyset = \{\}$  is the empty set (i.e., the set with no elements). A few more examples, or a simple induction argument, would quickly verify that, if<sup>4</sup>  $\text{card}(X) = n$ , then  $\text{card}(\mathcal{P}(X)) = 2^n$ . This leads immediately to Cantor's famous theorem.

**Theorem 2.4.** *Let  $X$  be any nonempty set and let  $\mathcal{P}(X)$  denote the power set of  $X$ . Then  $\text{card}(X) < \text{card}(\mathcal{P}(X))$ . In other words, the power set of  $X$  is always larger than  $X$  itself.*

Before proving this theorem, we note that the idea behind the proof is an ingenious self-referencing trick. In fact, this kind of self-reference revealed some subtle flaws in early attempts at axiomatizing set theory.

Consider the statement, "I am the man who cuts any man's hair who does not cut his own." We then ask, "Does this person cut their own hair?" Their statement claims that if they cut their own hair, then they must not cut their own hair. On the other hand, if they refuse to cut their own hair, then they must! This kind of self-referencing paradoxical statement is known as Russell's paradox.

In fact, this kind of statement reveals why one cannot consider the collection of *all* sets to be a set. Why? Suppose that  $\mathcal{U}$  is some set. From this we build a new set  $R_S = \{x \in \mathcal{U} \mid x \notin x\}$  ( $R_S$  for Russell's set). Notice that  $R_S$  is the set of everything that belongs to  $\mathcal{U}$  such that it does not belong to itself. Confused yet? Anyway, suppose  $R_S \in R_S$ . Then by definition, we would have that  $R_S$  is something in  $\mathcal{U}$  such that  $R_S \notin R_S$  (this is the definition of the set  $R_S$ ). However, this contradicts the supposition that  $R_S \in R_S$ . Therefore, we must conclude that  $R_S \notin R_S$ . Notice that we again seem to be in trouble. By definition if  $R_S \notin R_S$  and  $R_S \in \mathcal{U}$ , then  $R_S \in R_S$  (another contradiction). Thus we must conclude that  $R_S \notin \mathcal{U}$ . What we have just shown is that no matter what set you start with (i.e.,  $\mathcal{U}$ ), there will always be some new set that is missed (e.g.,  $R_S$ ). Therefore, one cannot speak of a "set of all sets" – no such thing exists.

Now let us return to the proof of Cantor's theorem. We will note that this proof relies on essentially this same trick:

**Proof:** (Cantor's Theorem) Suppose that  $X$  is some set.

First, clearly  $g : X \rightarrow \mathcal{P}(X)$  defined by  $g(x) = \{x\}$  is an injection so that  $\text{card}(X) \leq \text{card}(\mathcal{P}(X))$ . We now show that equality is impossible.

Suppose we have that  $f : X \rightarrow \mathcal{P}(X)$  is an injection between  $X$  and its power set. Consider the self-referencing trick set  $B = \{x \in X \mid x \notin f(x)\}$ . We note that  $f(x) \in \mathcal{P}(X)$  so that  $f(x)$  is a subset of  $X$  and thus " $x \notin f(x)$ " is actually a sensible thing to write down.

Now  $B$  is a subset of  $X$ , so  $B \in \mathcal{P}(X)$ . We can ask, "Is  $B$  in the range of  $f$ ?" Let us suppose that it is. Then there is some  $y \in X$  such that  $f(y) = B$ . Now notice that  $y \in B = f(y)$  if and only if  $y \notin f(y) = B$ . In words,  $y$  only belongs to the set  $B = f(y)$  if it doesn't! Therefore, we must conclude that there is no such  $y$ . Thus no element of  $X$  maps to  $B$ .

We have just shown that any injection between  $X$  and  $\mathcal{P}(X)$  must fail to be a surjection. Thus  $\text{card}(X) \neq \text{card}(\mathcal{P}(X))$ . In other words, we have shown that  $\text{card}(X) < \text{card}(\mathcal{P}(X))$ . ♦

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<sup>4</sup>Again, to be technically accurate, we mean that  $\text{card}(X) = \text{card}(\{1, 2, \dots, n\})$  and  $\text{card}(\mathcal{P}(X)) = \text{card}(\{1, 2, 3, \dots, 2^n\})$

Cantor's theorem tells us that for any set  $X$ :  $\text{card}(X) < \text{card}(\mathcal{P}(X)) < \text{card}(\mathcal{P}(\mathcal{P}(X))) < \dots$  and so, in particular,  $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) < \text{card}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \dots$ . Since  $\text{card}(\mathbb{N}) = \aleph_0$  is countable infinity,  $\text{card}(\mathcal{P}(\mathbb{N}))$  must be an uncountable infinity. This also establishes that there are infinitely many distinct infinite cardinalities and that no matter how big our set is, its power set is even larger. What is truly intriguing – as we have already seen – is that some collections are too large to be sets! There is no such thing as the “set of all sets”. We saw that assuming the existence of such an object leads to a contradiction. Wait! Did you see it? If  $\aleph_0 = \text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N}))$ , then there must be something larger than  $\aleph_0$ . Let us now use Cantor's diagonalization argument to show that  $\mathbb{R}$  is uncountable (i.e.,  $\text{card}(\mathbb{N}) < \text{card}(\mathbb{R})$ ).

**Theorem 2.5.** *The set of real numbers is uncountable.*

**Proof:** For sake of contradiction, suppose that  $\mathbb{R}$  is countable. Therefore, every subset of  $\mathbb{R}$  is countable and in particular the interval,  $I = [0, 1)$ , is countable. This means we can list the elements of  $I$  say  $I = \{x_1, x_2, x_3, \dots\}$ .

Next, every real number has a decimal expansion. In fact, it has a unique decimal expansion if we do not allow trailing 9's (for example,  $12.3\bar{9} = 12.4$ ). Consider  $x_j \in I$  so that  $0 \leq x_j < 1$ . Expand  $x_j$  (without trailing 9's) and get  $x_j = 0.d_{j1}d_{j2}d_{j3}\dots = d_{j1}10^{-1} + d_{j2}10^{-2} + d_{j3}10^{-3} + \dots$  where each digit  $d_{ij} \in \{0, 1, \dots, 9\}$ . Now focus on the  $i^{\text{th}}$  decimal digit of the  $i^{\text{th}}$  number in our list:

$$\begin{aligned} x_1 &= 0.\overline{d_{11}}d_{12}d_{13}d_{14}\dots \\ x_2 &= 0.d_{21}\overline{d_{22}}d_{23}d_{24}\dots \\ x_3 &= 0.d_{31}d_{32}\overline{d_{33}}d_{34}\dots \\ &\vdots \end{aligned}$$

Define  $y_i$  to be 1 if  $d_{ii} = 0$  and  $y_i = d_{ii} - 1$  otherwise. Then  $y = 0.y_1y_2y_3\dots$  is a real number in  $I = [0, 1)$ . We chose  $y$ 's digits so that it does not end in trailing 9's (this also keeps us from getting  $0.\bar{9} = 1$ ).

Notice that  $y \neq x_i$  for each  $i = 1, 2, \dots$  since  $y$  and  $x_i$  have differing  $i^{\text{th}}$  digits. Thus  $y$  is not on the list and so our list is incomplete (contradiction). ♦

Another way to establish that  $\mathbb{R}$  is uncountable is to show  $\text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N}))$ .<sup>5</sup> This again shows that  $\mathbb{R}$  is strictly larger than  $\mathbb{N}$ . We call the cardinality of the real numbers *continuum* and denote it by  $\text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}$ .

Again, recall that  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  is the set of integers and since we can list them off:  $0, -1, 1, -2, 2, -3, 3, \dots$  we know that  $\mathbb{Z}$  is countable. As we have seen,  $\mathbb{Q}$  is also countable, as we already listed them off. In fact, from our enumeration of  $\mathbb{Q}$  we may have already anticipated that  $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(p, q) \mid p, q \in \mathbb{Z}\}$  is countable:  $\text{card}(\mathbb{Z}^2) = \text{card}(\mathbb{N})$ . To show this concretely, plot the elements of  $\mathbb{Z}^2$  as grid points in the plane and then list them off by spiraling outward.

<sup>5</sup>Here we sketch a proof: First, one can find an invertible function between  $\mathbb{R}$  and the interval  $I = [0, 2)$  so that  $\mathbb{R}$  and  $I$  have the same cardinality. Next, each element of  $b \in I$  can be represented in a binary expansion:  $b = b_0.b_1b_2\dots = b_02^0 + b_12^{-1} + b_22^{-2} + \dots$  each binary digit  $b_i$  being either 0 or 1. Rule out expansions ending in repeated 1's (such as  $0.01011\bar{1}$ ). Create a set  $B = \{k \in \mathbb{N} \mid b_k = 1\}$ . So each element of  $I$  is associated with a subset of  $\mathbb{N}$ . This gives us an injection from  $I$  to  $\mathcal{P}(\mathbb{N})$ . Conversely take a subset  $B$  of  $\mathbb{N}$  and construct a real number whose  $(2i)^{\text{th}}$ -digit is 1 if  $i \in B$  and 0 otherwise. By making every other digit zero we have avoided trailing 1's. One can show this is an injection from  $\mathcal{P}(\mathbb{N})$  to  $I$ . Using the Cantor-Schröder-Berstein theorem, we get that there is an invertible map between  $I$  and  $\mathcal{P}(\mathbb{N})$ .



For infinite sets this is no longer true. Consider the natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$ . We can order this set in infinitely many distinct ways. For example, the standard order  $0, 1, 2, \dots$  behaves differently than the ordering  $\dots, 5, 3, 1, 0, 2, 4, \dots$  (here we list the odd numbers backwards then followed by the even numbers). The first ordering has a first but not a last element while the other ordering has neither a first nor a last element. But how about  $0, 2, 4, \dots, 1, 3, 5, \dots$  (the even numbers followed by the odd numbers)? This is quite different from our first (natural) ordering. Notice that in the natural ordering 0 is the only element not *immediately* preceded by any other element. However, in this new ordering both 0 and 1 are not *immediately* preceded by any other element. We could also consider  $1, 2, 3, \dots, 0$  (all positive numbers then followed by zero). This ordering has both a first and last element. So does  $2, 3, 4, \dots, 0, 1$  (all numbers two or larger then followed by zero and one) but this time the final element has an immediate predecessor. These examples should illustrate that there are infinitely many ways to organize the order structure of an infinite set. Let us get a little more technical.

**Definition 3.1.** Consider a set  $X$  and define a relation  $\leq$  on  $X$  so that for any two elements  $a, b \in X$  we either have  $a \leq b$  or not. Assume the following properties hold (for any  $a, b, c \in X$ ):

**Reflexive:** We always have  $a \leq a$ .

**Antisymmetric:** If  $a \leq b$  and  $b \leq a$ , then  $a = b$

**Transitive:** If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ .

Such a relation,  $\leq$ , is called a partial order on  $X$  ( $X$  is a “poset”).

If in addition, we have that for every  $a, b \in X$  either  $a \leq b$  or  $b \leq a$ , then we call the relation,  $\leq$ , a total order on  $X$ .

From this we fix related notations as follows:  $a < b$  means  $a \leq b$  but  $a \neq b$  and  $a \geq b$  means  $b \leq a$  etc. As a consequence, if  $X$  is totally ordered by  $\leq$ , then for all  $a, b \in X$  exactly one of the following holds:  $a < b$ ,  $a = b$ , or  $a > b$  (trichotomy).

Our usual  $\leq$  on  $\mathbb{R}$  is a total order. This is also true for  $\mathbb{Q}$ ,  $\mathbb{Z}$ , and  $\mathbb{N}$ . On the other hand, notice that for sets  $A, B$ , and  $C$  we always have that  $A \subseteq A$  (reflexive), if  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$  (antisymmetric), and if  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$  (transitive). This means that given a collection of sets, the subset relation,  $\subseteq$ , partially orders that collection of sets. But generally this is not a total order. For example, consider  $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$  (the power set of  $\{1, 2\}$ ). Notice that both  $\{1\} \not\subseteq \{2\}$  and  $\{2\} \not\subseteq \{1\}$ , so we fail to have a total order. Before getting to ordinals, we need one more ordering concept.

**Definition 3.2.** Let  $X$  be totally ordered by  $\leq$ . We say that  $X$  is well ordered by  $\leq$  if every nonempty subset of  $X$  has a least element: if  $A \subseteq X$  and  $A \neq \emptyset$ , there is some  $a \in A$  such that for all  $x \in A$  we have  $a \leq x$ .

Notice that while  $\mathbb{R}$  and  $\mathbb{Z}$  are totally ordered, they are not well ordered (they have no least elements). On the other hand,  $\mathbb{N} = \{0, 1, 2, \dots\}$  is, not just totally ordered but is in fact, well ordered by  $\leq$ . In fact, this is usually referred to as the *Well Ordering Principle*. This property of the natural numbers is equivalent to the principle of mathematical induction (each principle easily implies the other) [Birkhoff, et al., 1967].

Thus the natural numbers come with a built-in well ordering. We now turn this around and ask the question, is there a way to well order any arbitrary set? If one accepts the Axiom of Choice, the answer is yes.



Loosely speaking, the Axiom of Choice states that one can select an element from each set in a (potentially infinite) collection of nonempty sets. More formally, if  $A_i$  (where  $i \in I$  for some index set  $I$ ) is a collection of nonempty sets, the Axiom of Choice asserts the existence of a function  $f : I \rightarrow \bigcup_{i \in I} A_i$  where for each  $i \in I$  we have  $f(i) \in A_i$ . This can be restated as follows: a cartesian product of nonempty sets,  $\times_{i \in I} A_i$ , is nonempty. The Axiom of Choice is often presented as a statement called Zorn's Lemma. It is also equivalent to statements that look rather harmless like: every vector space has a basis [Mendelson, 1964] and [Birkhoff, et al., 1967].

Most relevant to us, the Axiom of Choice is equivalent to stating that every set has a well ordering. For countable sets, this is fairly obvious. A countable set can be put into a one-to-one correspondence with a subset of  $\mathbb{N}$ . We can then use that correspondence to transport the counting numbers' ordering to our countable set. Things get weird when we turn our attention to larger sets. Consider the real numbers. While  $\mathbb{R}$  is totally ordered by  $\leq$ , this is not a well ordering (there is no minimum real number).<sup>7</sup>

We are now ready to build the collection of ordinal numbers. (As previously stated, some collections are too large to be sets. The ordinals form such a collection.) First, following John von Neumann, we will build a concrete model of the natural numbers from set theory. Let  $0 = \emptyset$ . We are not saying that secretly the number zero is the empty set. We are merely using the empty set as a model for how we want zero to work. Next, let  $1 = \{0\} = \{\emptyset\}$ , so 1 is the set containing the empty set.<sup>8</sup> Next, let  $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$ . In general, let  $n + 1 = n \cup \{n\}$ . For example,  $4 = 3 + 1 = 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\} = \{0, 1, 2, 3\}$ . In other words,  $n = \{0, 1, \dots, n-1\}$ . It is important to note that we are now employing  $n$  as both a ordinal number and as defining a set. This is a neat trick. We've built our numbers from literally nothing!

This construction has several nice features. First, we have that  $m \leq n$  if and only if  $m \subseteq n$  (as sets). Also,  $n \cup m = \max\{n, m\}$  and  $n \cap m = \min\{n, m\}$ . Finally, notice that  $\text{card}(n) = n$ . This construction gives a collection of sets which (after defining addition and multiplication) satisfy Giuseppe Peano's axioms describing the natural numbers.

We can now create a new ordinal from an old one by the process of creating a successor:  $s + 1 = s \cup \{s\}$ . We can also create an ordinal by getting a union of a (potentially infinite) collection of ordinals. If this is a finite union of ordinals, we have  $n_1 \cup \dots \cup n_k = \max\{n_1, \dots, n_k\}$ . Notice that  $0 \cup 1 \cup 2 \cup \dots = \mathbb{N}$ . Thus  $\mathbb{N}$  is an ordinal. To indicate we are thinking of  $\mathbb{N}$  not as the set of natural numbers but as the first infinite ordinal, we write  $\omega = \mathbb{N}$ , where  $\omega$  represents the set of natural numbers together with order structure. Then  $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \dots, \omega\}$ . Likewise, we can create  $\omega + 1$ ,  $\omega + 2$ , etc. We even have  $\omega \cdot 2 = \omega + \omega = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots\}$ . Notice that  $\omega + \omega$  has the same order structure as one of our motivating ordering examples:  $0, 2, 4, \dots, 1, 3, 5, \dots$  (all even numbers followed by all odd numbers).

We should distinguish between two kinds of ordinals. A *successor ordinal* is an ordinal,  $\beta$ , of the form  $\beta = \kappa + 1$  for some ordinal  $\kappa$ . Ordinals such as 5 and 123 and  $\omega + 2$  are successor ordinals since they are immediate successors of some other ordinal:  $5 = 4 + 1$ ,  $123 = 122 + 1$ ,  $\omega + 2 = (\omega + 1) + 1$ . If an ordinal is not a successor, it is called a *limit ordinal*;  $\omega$  and  $\omega + \omega$  are limit ordinals. We cannot find some  $\kappa$  such that  $\kappa + 1 = \omega$  because  $\kappa$  would have to be  $\omega$  with

<sup>7</sup>This cannot be fixed by simply moving one number out of place. In fact, every open interval,  $I = (a, b)$ , is a nonempty subset of  $\mathbb{R}$ , but none of these intervals contains a minimum element since  $a \notin I$ . Still, the Axiom of Choice states that  $\mathbb{R}$  *can be* well ordered. It just turns out that such an order cannot be written down in a concrete way. In fact, Paul Cohen showed that it is consistent with Zermelo-Fraenkel set theory (i.e., standard set theory without the Axiom of Choice) that there is no well ordering on the reals [Kanamori, 2008]!

<sup>8</sup>This often confuses students:  $\emptyset = \{\}$  is empty while  $\{\emptyset\} = \{\{\}\}$  is not, since it contains one thing:  $\emptyset$ .

the “last” natural number removed. Likewise,  $\omega + \omega \neq \kappa + 1$  for any  $\kappa$ . We only get ordinals like  $\omega$  by creating a union of an infinite collection of ordinals:  $\omega = \bigcup_{n \in \mathbb{N}} n$ ;  $\omega$  cannot be the result of performing a “successor” operation on some ordinal.

Therefore, all ordinals come from the operations of successor and unioning ordinals. In fact, given a well ordered set  $X$ , there is a unique ordinal  $\kappa$  and invertible function  $f : X \rightarrow \kappa$  such that for all  $a, b \in X$  we have  $a \leq b$  (in  $X$ ) if and only if  $f(a) \leq f(b)$  (in  $\kappa$ ). Thus, every well ordered set can be associated with a unique ordinal number. This means that each (well ordered) order structure is modeled by a unique ordinal. For example,  $3 = \{0, 1, 2\}$  models the order structure of “first, second, last”. We can now use this understanding to consider some arithmetic in the ordinals.

We begin with addition. Take two ordinals,  $\alpha$  and  $\beta$ . Create a new set  $\alpha \cup \beta'$  where  $\beta'$  is a disjoint copy of  $\beta$  (just take each element of  $\beta$  and add a prime to make it different). Give  $\alpha \cup \beta'$  the natural ordering where we use  $\alpha$ ’s ordering for elements drawn from  $\alpha$  and  $\beta$ ’s ordering for elements from  $\beta'$  and declare that everything in  $\alpha$  comes before everything in  $\beta'$ . For example,  $2 \cup 3'$  would be ordered:  $0 < 1 < 0' < 1' < 2'$ . Finally, it can be shown that  $\alpha \cup \beta'$  with this ordering is well ordered and thus corresponds to a unique ordinal number. We define  $\alpha + \beta$  to be that ordinal. For example,  $2 + 3$  is the ordinal associated with  $0 < 1 < 0' < 1' < 2'$  which corresponds to the order structure  $0 < 1 < 2 < 3 < 4$ . Thus  $2 + 3 = 5$ . We finally know why  $1 + 1 = 2$ .

Here is where things get interesting. Consider  $3 + \omega$ . This would give  $0 < 1 < 2 < 0' < 1' < 2' < 3' < \dots$  which is the same order structure as  $\omega$ . Therefore,  $3 + \omega = \omega$ . But  $\omega + 3$  is ordered as  $0 < 1 < 2 < \dots < \omega < \omega + 1 < \omega + 2$ , so  $\omega + 3 \neq \omega$ . This means that ordinal addition is not commutative:  $3 + \omega \neq \omega + 3$ .

Next, we multiply. Again, let  $\alpha$  and  $\beta$  be ordinals. Recall that each ordinal number is a set, so we can consider the Cartesian product  $\beta \times \alpha$ . We give  $\beta \times \alpha$  an ordering as follows:  $(b_1, a_1) \leq (b_2, a_2)$  if either  $b_1 \leq b_2$  (in  $\beta$ ) or  $b_1 = b_2$  and  $a_1 \leq a_2$  (in  $\alpha$ ). In other words, we order lexicographically. Again,  $\beta \times \alpha$  is well ordered with such an ordering and thus is associated with a unique ordinal. We define  $\alpha \cdot \beta$  to be that ordinal.

For example, consider  $2 \cdot 3$ . The Cartesian product  $3 \times 2 = \{0, 1, 2\} \times \{0, 1\}$  is ordered as follows:  $(0, 0) < (0, 1) < (1, 0) < (1, 1) < (2, 0) < (2, 1)$ . This has the same order structure as  $0 < 1 < 2 < 3 < 4 < 5$  (i.e., 6) thus  $2 \cdot 3 = 6$ .

The product,  $2 \cdot \omega$  has the structure determined by the set  $\omega \times 2$  ordered as  $(0, 0) < (0, 1) < (1, 0) < (1, 1) < (2, 0) < \dots$  which is the same order type as  $\omega$  (so that  $2 \cdot \omega = \omega$ ). On the other hand,  $\omega \cdot 2$  corresponds to the set  $2 \times \omega$  ordered as  $(0, 0) < (0, 1) < (0, 2) < \dots < (1, 0) < (1, 1) < (1, 2) < \dots$ . This means  $\omega \cdot 2 = \omega + \omega$ . In general, think of  $\alpha \cdot \beta$  as  $\beta$  consecutive copies of  $\alpha$ . Thus  $\omega$  copies of 2 just yields  $\omega$  but 2 copies of  $\omega$  yields  $\omega + \omega$ . Since  $\omega = 2 \cdot \omega \neq \omega \cdot 2 = \omega + \omega$ , ordinal multiplication is also noncommutative.

Notice also, that while we have distributivity on the right:  $\omega \cdot 2 = \omega \cdot (1 + 1) = \omega \cdot 1 + \omega \cdot 1 = \omega + \omega$ , the distributive law does not work on the left:  $\omega = 2 \cdot \omega = (1 + 1) \cdot \omega \neq 1 \cdot \omega + 1 \cdot \omega = \omega + \omega$ . While some familiar rules work, like  $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$  and associativity, other rules fail. When adding and multiplying ordinals, we need to proceed with caution.

There is another approach to ordinal arithmetic, namely recursion, which mathematicians in this field commonly use. For addition, fix the base case:  $\alpha + 0 = \alpha$ . For successors like  $\beta + 1$ , define  $\alpha + (\beta + 1)$  as the successor of  $\alpha + \beta$  (i.e.,  $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ ). Finally, for limit ordinals such as  $\beta$ , define  $\alpha + \beta = \bigcup_{\gamma \in \beta} (\alpha + \gamma)$ . With addition established, multiplication

is then built in a similar way from addition and recursion. Base case:  $\alpha \cdot 0 = 0$ . Successors:  $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ . Limit ordinals:  $\alpha \cdot \beta = \bigcup_{\gamma \in \beta} \alpha \cdot \gamma$ . One can also define exponentiation from multiplication and recursion. Base case:  $\alpha^0 = 1$ . Successors:  $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ . Limit ordinals:  $\alpha^\beta = \bigcup_{\gamma \in \beta} \alpha^\gamma$ .

We could have defined exponentiation,  $\alpha^\beta$ , by placing a lexicographic order certain order on a set of functions from  $\beta$  to  $\alpha$  (functions of “finite support”) and associating with an ordinal. However, let’s not concern ourselves with this interpretation. Exponentiation like addition and multiplication can be weird. For example,  $(\omega \cdot 2)^2 = (\omega \cdot 2) \cdot (\omega \cdot 2) = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega \cdot \omega \cdot 2 = \omega^2 \cdot 2 = \omega^2 + \omega^2$ . On the other hand,  $\omega^2 \cdot 2^2 = \omega^2 \cdot 4 = \omega^2 + \omega^2 + \omega^2 + \omega^2$  which is strictly bigger than  $\omega^2 + \omega^2$ . Thus  $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$ . As one last bizarre example, we state without proof that  $2^\omega = \omega$ . Ordinal arithmetic can be very strange.

## 4 Connecting Ordinals with Cardinals

With cardinal and ordinal numbers in hand, we can now give an existence to our currently disembodied notion of “ $\text{card}(X)$ ”. We do so by connecting the two types of numbers.

It turns out that the collection of all ordinals is well ordered in the sense that any non-empty subcollection of ordinals contains a least ordinal (remember that ordinals are ordered by containment:  $\alpha \leq \beta$  if and only if  $\alpha \subseteq \beta$ ). If we consider all of the ordinals of a particular cardinality this is a non-empty collection, thus there must be a smallest such ordinal (smallest in the ordinal sense).<sup>9</sup> We call such an ordinal a *cardinal number*. There are ordinals of every cardinality, so given a set  $X$ , there is at least one ordinal  $\alpha$  such that  $\text{card}(X) = \text{card}(\alpha)$ . Consider all of the ordinals with cardinality  $\text{card}(X)$  and suppose that  $\alpha$  comes first (to be clear,  $\alpha$  is the first ordinal in the ordinal ordering whose cardinality matches that of  $X$ ), then we say  $\text{card}(X) = \alpha$ .

This can be quite confusing mostly because when sets are finite, there is only one ordinal per cardinality. For example, the only ordinal with three elements is 3 itself! We write  $\text{card}(\{a, b, c\}) = 3$ . On the other hand,  $\omega + 5$ ,  $\omega^2$ ,  $\omega + \omega$  are all countable ordinals. But we also know that each collection of ordinals of the same cardinality must have a least element. For instance, while  $\omega$ ,  $\omega + 17$ ,  $\omega \cdot 3$ , and  $\omega^2$  all have the same cardinality, as ordinals,  $\omega < \omega + 17 < \omega \cdot 3 < \omega^2$ . The smallest countable ordinal is  $\omega$  itself thus  $\text{card}(\omega) = \text{card}(\omega + 17) = \text{card}(\omega \cdot 3) = \text{card}(\omega^2) = \aleph_0$ .

When thinking of  $\mathbb{N} = \omega = \{0, 1, 2, \dots\}$  as a cardinal number, we call it  $\aleph_0$ . Our three labels  $\mathbb{N}$ ,  $\omega$ , and  $\aleph_0$  represent the same collection of objects, but signify a different structure being considered (i.e., set versus ordinal versus cardinal). So  $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \aleph_0$  (this is countable infinity). The first uncountable ordinal (thought of as a cardinal) is called  $\aleph_1$ .

Cardinal arithmetic is much easier than ordinal arithmetic. We define addition as follows:  $\text{card}(A) + \text{card}(B)$  is the cardinality of the disjoint union of  $A$  and  $B$ . It turns out that the cardinality of the union of two disjoint sets is the same as the larger of the two sets (as long as at least

<sup>9</sup>Note that we are considering two kinds of smallness/order. We order ordinals one way and cardinals another. It turns out that order on cardinal numbers is a sampling of the order on ordinals (all cardinal numbers are ordinal numbers but not vice-versa). If we consider cardinalities of ordinals, we see that  $\alpha \leq \beta$  (as ordinals) implies  $\text{card}(\alpha) \leq \text{card}(\beta)$ . But  $\text{card}(\alpha) = \text{card}(\beta)$  does not mean  $\alpha = \beta$  (like we’re used to for finite ordinals). For example,  $\omega + 1$  and  $\omega$  have equal cardinalities but are not the same ordinal.

one of the sets is infinite), so for infinite cardinals  $\text{card}(A) + \text{card}(B) = \max\{\text{card}(A), \text{card}(B)\}$ . Thus  $\aleph_1 + 5$ ,  $\aleph_1 + \aleph_0$ , and  $\aleph_1 + \aleph_1$  are all just  $\aleph_1$ .

Cardinal multiplication is also easy. We define  $\text{card}(A) \cdot \text{card}(B) = \text{card}(A \times B)$  (the cardinality of the Cartesian product of  $A$  and  $B$ ). Again, as long as at least one of our sets is infinite (and the other isn't empty) we have  $\text{card}(A) \cdot \text{card}(B) = \max\{\text{card}(A), \text{card}(B)\}$  since in the context of infinite sets the Cartesian product's cardinality is just the cardinality of the bigger set. For example,  $2\aleph_0$  and  $\aleph_0 \cdot \aleph_0$  are both just  $\aleph_0$ . In summary, for nonzero cardinals  $\alpha$  and  $\beta$  where at least one is infinite, we have that  $\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}$ .

Finally, exponentiation is also simple to define:  $\text{card}(A)^{\text{card}(B)} = \text{card}(A^B)$  where  $A^B$  is the set of all functions  $f : B \rightarrow A$  (functions from  $B$  to  $A$ ).

Consider a set  $X$  and let  $A \subseteq X$ . We can define a characteristic function  $\chi_A : X \rightarrow 2$  by  $\chi_A(x) = 1$  if  $x \in A$  and  $\chi_A(x) = 0$  if  $x \notin A$ . Thus  $\chi_A$  just tells us if an element of  $X$  belongs to  $A$  or not (think 1 means yes and 0 means no). Conversely, given a function  $f : X \rightarrow 2$  we can let  $A = \{x \in X \mid f(x) = 1\}$ . In fact, if so, we have that  $f = \chi_A$ . So we can see that every subset of  $X$  is uniquely associated with a function from  $X$  to 2 (i.e., an element of  $2^X$ ). For instance, consider  $\mathcal{P}(\{a, b, c\})$ . Then the subset  $A = \{a, c\}$  corresponds with the characteristic function  $\chi_A(a) = \chi_A(c) = 1$  and  $\chi_A(b) = 0$ . Conversely, if  $f : \{a, b, c\} \rightarrow \{0, 1\}$  is the function  $f(a) = 1$  and  $f(b) = f(c) = 0$  then  $f = \chi_B$  where  $B = \{a\}$ . The function  $g(a) = g(b) = g(c) = 0$  is the characteristic function of the empty set:  $g = \chi_\emptyset$ . The  $2^3 = 8$  functions from  $\{a, b, c\}$  to  $\{0, 1\}$  correspond to the  $2^3$  subsets of  $\{a, b, c\}$ . Therefore,  $\text{card}(\mathcal{P}(X)) = \text{card}(2^X) = 2^{\text{card}(X)}$ . Recalling Cantor's theorem, we see that  $\text{card}(X) < 2^{\text{card}(X)}$ . We can also now see that  $\text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}$  (continuum). Notice that  $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} \dots$ <sup>10</sup> Indeed, therefore, there is an infinite number of infinities! Thanks, again, Buzz Lightyear for getting this right and communicating such so artfully.

Just as with addition and multiplication, exponentiation carries some oddities. For example, if we know that  $A \subseteq B$ , then there are at least as many functions from  $A$  to  $C$  as there are from  $B$  to  $C$ . Therefore, if  $\alpha \leq \beta$ , then  $\alpha^\gamma \leq \beta^\gamma$ . This implies that  $2^{\aleph_0} \leq 3^{\aleph_0} \leq (\aleph_0)^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0}$ . However,  $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$  because  $\aleph_0 \cdot \aleph_0 = \aleph_0$ , since maximum computes multiplication.<sup>11</sup> Therefore, our whole chain of inequalities must actually be equalities! This means that:

$$2^{\aleph_0} = 3^{\aleph_0} = \aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0}.$$

We will let the reader decide if this mathematical statement is weird or wonderful.

Since the real numbers are uncountable,  $\aleph_1 \leq 2^{\aleph_0}$  (recall that  $\aleph_1$  is the smallest uncountable cardinal). It is natural to ask if  $\aleph_1 = 2^{\aleph_0}$ ? This is known as the Continuum Hypothesis (CH). A wider sweeping conjecture came to be known as the Generalized Continuum Hypothesis (GCH). This asserts that  $2^{\aleph_n} = \aleph_{n+1}$  for all ordinals  $n$  where  $\aleph_{n+1}$  is next cardinal after  $\aleph_n$ . In other words, while Cantor tells us that the power set of  $X$  is larger than  $X$ , GCH states that there are no sizes in between. If CH were true,  $2^{\aleph_0} = \aleph_1$ , there would be no "sizes of infinity" between countable (i.e., the size of the natural numbers) and continuum (i.e., the size of the real numbers). If GCH were true, we would know exactly how cardinal exponentiation works (each exponential just moves us up to the next cardinal).

<sup>10</sup>Some authors use  $\beth_n$  (Hebrew letter Beth) to represent the cardinality of the  $n^{\text{th}}$  powerset of  $\mathbb{N}$  so  $\beth_0 = \aleph_0 = \text{card}(\mathbb{N})$ ,  $\beth_1 = 2^{\aleph_0} = \text{card}(\mathcal{P}(\mathbb{N}))$ ,  $\beth_2 = 2^{\beth_1} = 2^{2^{\aleph_0}}$  etc.

<sup>11</sup>Functions  $\aleph_0 \rightarrow (\aleph_0 \rightarrow 2)$  (functions from  $\aleph_0$  to functions from  $\aleph_0$  to 2) are essentially just functions  $\aleph_0 \times \aleph_0 \rightarrow 2$  (functions from ordered pairs from  $\aleph_0 \times \aleph_0$  to 2) essentially:  $f(x, y) = f_x(y)$ . Thus,  $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0}$ .

David Hilbert in his famous address to the International Congress of Mathematicians in 1900 gave a list of important open problems. The Continuum Hypothesis was the very first problem on his list. It came as a great surprise to mathematicians that CH cannot be proven or disproven within the context of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC, i.e., standard set theory). In particular, in 1940 Kurt Gödel showed inside of ZFC one cannot disprove CH. On the other hand, in the 1963 Paul Cohen showed that inside ZFC one cannot prove CH either! In other words, either CH or its negation could be added to ZFC as a new and independent axiom. So in the context of standard set theory, we have no hope of truly understanding cardinal exponentiation – or *exactly* how big the real numbers really are! For example, Cohen showed that it is consistent with ZFC to assume that  $2^{\aleph_0}$  is  $\aleph_1$  or  $\aleph_2$  or  $\aleph_{\omega^2+1}$  or almost anything one might wish<sup>12</sup> [Cohen, 1964].

## 5 Limiting

Notice that in the entire text above regarding cardinal and ordinal numbers – even when representing countable and uncountable infinite notions – the symbol  $\infty$  was never used. This is because  $\infty$  provides a third context for infinity.

Students informally encounter  $\infty$  as early as in elementary grades. They then encounter it more formally in precalculus and calculus in the context of limits. Note that, rather than calculating  $f(\infty)$ , we typically only claim to approach  $\infty$  through a limit (i.e.,  $\lim_{x \rightarrow \infty} f(x)$ ) rather than actually hit  $\infty$ . While not totally unrelated, this notion of infinity is not the same as  $\omega$ ,  $\aleph_0$ ,  $2^{\aleph_0}$ , or  $\aleph_1$ .

In many ways,  $\infty$  denotes a location, a place we are heading towards, rather than a size. Often instructors and authors emphasize that we can approach but never get to  $\infty$ . As discussed in [Cook et al., 2020], this does not have to be the case. One can define both  $\infty$  and/or  $-\infty$  to be an extended kind of real number and then consider either one- or two-point compactifications of the real numbers:  $\mathbb{R} \cup \{\infty\}$  or  $\mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$ . Here we can treat infinity much like any other real number and make sense out of arithmetic operations and function evaluations as long as we are sufficiently careful.

As with cardinal and ordinal numbers, some arithmetic is afforded us in respect to  $\infty$ . For instance, for  $a \in \mathbb{R}$  and  $a > 0$ :  $\infty + a \cdot \infty = \infty$ ,  $\infty^a = \infty$ , and  $\infty/a = \infty$ . However,  $\infty - \infty$  and  $\infty/\infty$  are ill defined.

The brevity of our consideration of  $\infty$  in this context is due to the respective commonality with which it is encountered by students and our discussion in [Cook et al., 2020].

## 6 Investigations

In this section, we provide some investigations appropriate to high school and undergraduate students who wish to challenge themselves and understand more of the concepts of infinity.

1. For both cardinal and ordinal numbers and for both the operations of addition and multiplication, determine which properties hold and provide counter examples if they fail: commutativity, associativity, and distributivity.

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<sup>12</sup>Regarding this point, in informal mathematical parlance we interject the commentary, “Wow.”

2. Explain why the collection of polynomials with integer coefficients has a cardinality of  $\aleph_0$ . Use this to explain why the set of Algebraic Numbers must also have a cardinality of  $\aleph_0$ .
3. Give examples of sets which are: (a) totally ordered; (b) well ordered; (c) totally ordered but not well ordered.
4. Find at least five resources discussing the Axiom of Choice and summarize in your own words.
5. In the paper, we wrote: “We can place an equivalence relation on the collection of ordinals as follows:  $\alpha \sim \beta$  if and only if  $\text{card}(\alpha) = \text{card}(\beta)$ . . . Every finite ordinal is a cardinal and distinct finite ordinals have distinct cardinalities. . . Each equivalence class of ordinals of the same cardinality must have a least element. . . We call the smallest such ordinal a cardinal number.” Rewrite these sentences in your own words with your own examples.

## 7 Conclusion

We have considered three contexts for the notion of infinity: cardinal number ( $\aleph_0$  and  $2^{\aleph_0}$ ); ordinal number ( $\omega$  and  $\omega + \omega$ ); and as a destination ( $\infty$ ). Most high school students have only had experience with the latter. We hope that this brief consideration has opened new vistas regarding infinity and stimulates interest in these wonderful notions.

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